

A higher chromatic analogue of the image of J

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Abstract

We prove a higher chromatic analogue of Snaith's theorem which identifies the K-theory spectrum as the localisation of the suspension spectrum of $\mathbb{C}P^\infty$ away from the Bott class; in this result, higher Eilenberg-MacLane spaces play the role of $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$. Using this, we obtain a partial computation of the part of the Picard-graded homotopy of the $K(n)$ -local sphere indexed by powers of (a shift of) the Gross-Hopkins dual of the sphere. Our main technical tool is a $K(n)$ -local notion generalising complex orientation to higher Eilenberg-MacLane spaces. As for complex-oriented theories, such an orientation produces a one-dimensional formal group law as an invariant of the cohomology theory. As an application, we prove a theorem that gives evidence to the chromatic redshift conjecture.

1 Introduction

The stable homotopy groups of a space X are defined as the colimit

$$\pi_j^S(X) = \lim_{m \rightarrow \infty} \pi_{j+m}(\Sigma^m X) = \lim_{m \rightarrow \infty} \pi_j(\Omega^m \Sigma^m X) = \pi_j(QX).$$

where $QX = \varinjlim \Omega^m \Sigma^m X$.

The J-homomorphism $J : \pi_j(O) \rightarrow \pi_j^S(S^0)$ may be regarded as a first approximation to the stable homotopy groups of S^0 ; here O denotes the infinite orthogonal group $\varinjlim O(m)$. It is induced in homotopy by the limit over m of maps

$$J_m : O(m) \rightarrow \Omega^m S^m,$$

where for a matrix $M \in O(m)$ regarded as a linear transformation $M : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $J_m(M) = M \cup \{\infty\} : S^n \rightarrow S^n$. There is an analogous function from the infinite unitary group U ; $J_U : U \rightarrow QS^0$ is given by composition with the forgetful map $U \rightarrow O$. The homotopy groups of the domains are computable via Bott periodicity, and are

$j \bmod 8$	0	1	2	3	4	5	6	7
$\pi_j(O)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	\mathbb{Z}	0	0	0	\mathbb{Z}
$\pi_j(U)$	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}

The work of Adams [Ada66] shows that for an odd prime p , in dimensions $3 \bmod 4$, the p -torsion of the image in $\pi_j^S(S^0)$ of the cyclic group $\pi_j(O)$ is isomorphic to \mathbb{Z}/p^{k+1} , when we can write $j+1 = 2(p-1)p^k m$, with m coprime to p . For other j which are $3 \bmod 4$, the p -torsion in the image of J is zero. If we are working away from $p = 2$, this computation may be done using U in place of O .

As this result indicates, the burden of computation in stable homotopy theory often encourages one to work locally at a prime p . That is, one computes the p -power torsion in $\pi_*^S(X)$, and then assembles the results together into an integral statement. One of the deeper insights of homotopy in the last half century is that one may, following Bousfield [Bou79], do such computations local to any *cohomology theory* E^* to get more refined computations. Morava's extraordinary K-theories $K(n)$ and E-theories E_n [Mor85] have proven particularly suited to this purpose; see, e.g., [MRW77, HS98].

When $n = 1$, $K(1)$ is identified with (a split summand of) mod p K-theory. The fact that $\pi_*(U) = \pi_{*+1}(K)$ for $* > 0$ suggests that Adams' computation of the p -torsion in the image of J is related to $K(1)$ -local homotopy theory. This is in fact the case; the localisation map

$$\pi_*^S(S^0) \rightarrow \pi_*(L_{K(1)}S^0)$$

carries im J isomorphically onto the codomain in positive degrees.

One substantial difference between the stable homotopy category and its $K(n)$ -local variant is the existence of exotic invertible elements. In the stable homotopy category, the only spectra which admit inverses with respect to the smash product are spheres; thus the *Picard group* of equivalence classes of such spectra is isomorphic to \mathbb{Z} . In contrast, the Picard group of the $K(n)$ -local category, Pic_n , includes p -complete factors as well as torsion (see, e.g., [HMS94, GHMR12]).

Our main result is a computation of part of the *Picard graded* homotopy of the $K(n)$ -local sphere $S := L_{K(n)}S^0$ analogous to the image of J computation. Throughout this paper p will denote an odd prime; when localising with respect to $K(n)$, the prime p is implicitly used.

Theorem 1.1. *Let $\ell \in \mathbb{Z}$, and write $\ell = p^k m$, where m is coprime to p . Then the group $[S\langle \det \rangle^{\otimes \ell(p-1)}, L_{K(n)}S^1]$ contains a subgroup isomorphic to \mathbb{Z}/p^{k+1} . Furthermore, if $n^2 < 2p - 3$, there is an exact sequence*

$$0 \rightarrow \mathbb{Z}/p^{k+1} \rightarrow [S\langle \det \rangle^{\otimes \ell(p-1)}, L_{K(n)}S^1] \rightarrow N_{k+1} \rightarrow 0$$

where $N_{k+1} \leq \pi_{-1}(S)$ is the subgroup of p^{k+1} -torsion elements.

This is proved as Corollary 5.10. Here, $S\langle \det \rangle \in \text{Pic}_n$ was introduced by Goerss et al. in [GHMR12]; it is defined below. When $n = 1$ and $p > 2$, $S\langle \det \rangle$ may be identified as $L_{K(1)}S^2$, and so this result recovers the constructive part of the classical image of J computation. More generally, $S\langle \det \rangle$ may be identified as a shift of the Brown-Comenetz dual of the n^{th} monochromatic layer of the sphere spectrum if $\max\{2n + 2, n^2\} < 2(p - 1)$ (see [HG94]).

1.1 The invertible spectrum $S\langle \det \rangle$

Morava's E -theories are Landweber exact cohomology theories E_n associated to the universal deformation of the Honda formal group Γ_n from \mathbb{F}_{p^n} to $\mathbb{W}(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$. When $n = 1$, E_1 is precisely p -adic K-theory. The Goerss-Hopkins-Miller theorem [GH04, GH05] equips the spectrum E_n with a continuous action of the *Morava stabiliser group*

$$\mathbb{G}_n := \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \ltimes \text{Aut}(\Gamma_n)$$

which lifts the defining action in homotopy. The work of Devinatz-Hopkins [DH04] and Davis [Dav06, Dav09] then allows one to define continuous¹ homotopy fixed point spectra with respect to this action in a consistent way. The associated homotopy fixed point spectrum is the $K(n)$ -local sphere: $E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S^0$.

¹All homotopy fixed point spectra considered in this article will be of the continuous sort.

The automorphism group $\text{Aut}(\Gamma_n)$ is known to be the group of units of an order of a rank n^2 division algebra over \mathbb{Q}_p ; the determinant of the action by left multiplication defines a homomorphism $\det : \mathbb{G}_n \rightarrow \mathbb{Z}_p^\times$. We will write $S\mathbb{G}_n$ for the kernel of this map. We may define the homotopy fixed point spectrum $E_n^{hS\mathbb{G}_n}$ for the restricted action of this subgroup.

This spectrum retains a residual action of $\mathbb{Z}_p^\times = \mathbb{G}_n/S\mathbb{G}_n$; for an element $k \in \mathbb{Z}_p^\times$, we will write the associated map as $\psi^k : E_n^{hS\mathbb{G}_n} \rightarrow E_n^{hS\mathbb{G}_n}$. The reader is encouraged to think of these automorphisms as analogues of Adams operations. Noting that $\mathbb{Z}_p^\times = \mu_{p-1} \times (1 + p\mathbb{Z}_p)^\times$ is topologically cyclic with generator $g = \zeta_{p-1}(1 + p)$, we define F_γ as the homotopy fibre of

$$\psi^g - \gamma : E_n^{hS\mathbb{G}_n} \rightarrow E_n^{hS\mathbb{G}_n}$$

for any $\gamma \in \mathbb{Z}_p^\times$. These spectra are always invertible, and in fact the construction $\gamma \mapsto F_\gamma$ defines a homomorphism $\mathbb{Z}_p^\times \rightarrow \text{Pic}_n$. When $\gamma = 1$, the associated homotopy fibre defines the homotopy fixed point spectrum for the action of \mathbb{Z}_p^\times , and so

$$F_1 = (E_n^{hS\mathbb{G}_n})^{h\mathbb{Z}_p^\times} \simeq E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S^0$$

In contrast, one defines $S\langle \det \rangle := F_g$.

1.2 A Snaith theorem

The K -theory spectrum admits a remarkable description due to Snaith [Sna79]. He shows that the natural inclusion of $\mathbb{C}P^\infty$ into $BU \times \mathbb{Z}$ localises to an equivalence

$$\Sigma^\infty \mathbb{C}P_+^\infty[\beta^{-1}] \simeq K.$$

Here the *Bott map* $\beta : S^2 \rightarrow \Sigma^\infty \mathbb{C}P_+^\infty$ is a reduced, stable form of the inclusion $\mathbb{C}P^1 \rightarrow \mathbb{C}P^\infty$.

Theorem 1.2. *There is a map $\rho_n : S\langle \det \rangle \rightarrow L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)_+$ and an equivalence of E_∞ -ring spectra*

$$L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)_+[\rho_n^{-1}] \rightarrow E_n^{hS\mathbb{G}_n}.$$

We will refer to the equivalent ring spectra of the theorem as R_n . One inverts the map ρ_n in the same fashion as for normal homotopy elements of a ring spectrum, via a telescope construction. A consequence of this theorem is that the homotopy of R_n is $S\langle \det \rangle$ -periodic. When $n = 1$, R_1 is simply p -adic K -theory, and this result is familiar as Bott periodicity. Indeed, the map ρ_1 is the $K(1)$ -localisation of β , and so we refer to ρ_n as a *higher Bott map*.

The proof of this theorem uses the computations of Ravenel-Wilson [RW80] of the Morava K -theories of Eilenberg-MacLane spaces, as well as the E_∞ obstruction theory developed by Goerss-Hopkins in [GH04]. It is proven as Corollary 3.20, below.

1.3 Higher orientation for $K(n)$ -local cohomology theories

The map β is traditionally used to define the notion of complex orientation for a cohomology theory; the associated formal group law is an important invariant of the cohomology theory, and central to the chromatic approach to homotopy theory. We use the map ρ_n to give a similar notion:

Definition 1.3. An n -orientation of a $K(n)$ -local ring spectrum E is a class $x \in E^{S\langle \det \rangle}(K(\mathbb{Z}_p, n+1))$ with the property that $\rho_n^*(x) \in E^{S\langle \det \rangle}(S\langle \det \rangle) = \pi_0 E$ is a unit.

Here, for an element $A \in \text{Pic}_n$ and a space X the notation $E^A(X)$ indicates the group

$$E^A(X) = [\Sigma^\infty X_+, A \otimes E];$$

this yields $E^m(X)$ when $A = S^m$. Grading the associated groups over all of Pic_n , we define

$$E^\star(X) := \bigoplus_{A \in \text{Pic}_n} E^A(X).$$

As for complex orientation, we show in Theorem 5.3 that an n -orientation of E gives a ring isomorphism $E^\star(K(\mathbb{Z}_p, n+1)) \cong E^\star[[x]]$. The multiplication in $K(\mathbb{Z}_p, n+1)$ then yields a formal group law $F(x, y) \in E^\star[[x, y]]$. An n -oriented cohomology theory is said to be *multiplicative* if its associated formal group law is of the form $F(x, y) = x + y + txy$ for a unit $t \in E^\star$.

Theorem 1.4. *R_n is the universal (i.e., initial) multiplicative n -oriented cohomology theory.*

This is Theorem 5.5 below. Our proof is modelled on an argument of Spitzweck-Østvær [SØ09] which gives a version of Snaith's theorem in the motivic setting.

This theorem allows us to identify \mathbb{Z}_p as a summand of $\pi_0(R_n)$, and hence of $[S\langle \det \rangle^{\otimes j}, R_n]$ for every j by Theorem 1.2. Furthermore, it implies that the action of the Adams operation ψ^g on this summand is via the j^{th} power of the identity character. Theorem 1.1 follows from these facts by the long exact sequence in $[S\langle \det \rangle^{\otimes j}, -]$ -groups associated to the fibre sequence

$$S \xrightarrow{\eta} R_n \xrightarrow{\psi^g - g} R_n$$

1.4 Algebraic K-theory and the chromatic redshift conjecture

For an A_∞ ring spectrum A , the space $\text{GL}_1(A)$ of units (the invertible components in $\Omega^\infty A$) admits a delooping $B\text{GL}_1(A)$. If A is in fact an E_∞ ring spectrum, its multiplication equips $B\text{GL}_1(A)$ with the structure of an infinite loop space. We will write $K(A)$ for the algebraic K -theory spectrum of A . There is a map of E_∞ ring spectra

$$i : \Sigma^\infty B\text{GL}_1(A)_+ \rightarrow K(A)$$

whose adjoint is given by the inclusion of A -lines into all cell A -modules (see [ABG⁺08]).

In [SW11], with Hisham Sati, we constructed an E_∞ map² $\varphi_n : K(\mathbb{Z}, n+1) \rightarrow \text{GL}_1 E_n$. Delooping once and composing with i , we obtain a map of E_∞ ring spectra

$$i \circ B\varphi_n : \Sigma^\infty K(\mathbb{Z}, n+2)_+ \rightarrow K(E_n).$$

We may localise both the domain and range of this map with respect to $K(n+1)$; it is natural to ask about the behaviour of the composite map

$$\beta_{n+1} = i \circ B\varphi_n \circ \rho_{n+1} : S\langle \det \rangle \rightarrow L_{K(n+1)} K(E_n)$$

(we note that the domain is the $K(n+1)$ -local $S\langle \det \rangle$).

Our methods are not suitable to construct such a map for the case $n = 0$ but it is instructive to consider this example. We interpret E_0 as singular cohomology with \mathbb{Q}_p coefficients; the resulting K -theory spectrum is the algebraic K -theory of \mathbb{Q}_p . The corresponding map $\beta_1 : S^2 \rightarrow L_{K(1)} K(\mathbb{Q}_p)$ should be considered as the image in the $K(1)$ -local category of the *Bott element* (considered in e.g., [Tho85] and [Mit97]) in the second p -adic algebraic K -theory of \mathbb{Q}_p .

²In the indicated paper, we constructed this map only at the prime 2. A mild generalisation of the argument is given for all primes in section 3.8 of this document. Indeed, φ_n is used in the equivalence given in Theorem 1.2.

Theorem 1.5. *For $p > 3$, multiplication by $\beta_{n+1} \in \pi_{S\langle \det \rangle} L_{K(n+1)} K(E_{n-1})$ is an equivalence. Therefore*

$$L_{K(n+1)} K(E_n) \simeq L_{K(n+1)} K(E_n) [\beta_{n+1}^{-1}],$$

and so the map $i \circ B\varphi_n$ makes $L_{K(n+1)} K(E_n)$ a nontrivial R_{n+1} -algebra spectrum.

This is proven as Corollary 6.3 below. It provides evidence for the chromatic redshift conjecture, as enunciated in Conjecture 4.4 of [AR06]. The spectrum of the theorem is the algebraic K-theory spectrum of a prominent $K(n)$ -local spectrum. After localisation at $K(n+1)$, we have shown it to support an algebra structure one chromatic level higher.

It is worth pointing out that this result is entirely $K(n+1)$ -local. The corresponding statement for $n = 0$ – that multiplication by the Bott element as considered by Thomason is a $K(1)$ -local equivalence – follows naturally from the fact that it descends to the familiar Bott class in the complex K-theory spectrum. A deeper statement in fact holds for $n = 0$: the Bott element exists prior to localisation, and multiplication by it is not nilpotent. Furthermore, inversion of this element defines the (smashing) $K(1)$ -localisation (see [Mit97]).

For $n = 1$, Ausoni has constructed in [Aus10] a class that he calls the *higher Bott element*, $b \in V(1)_{2p+2}(K(E_1))$, and has verified that it, too, is not nilpotent using topological cyclic homology techniques. Again, this construction occurs at a stage prior to $K(2)$ -localisation. We believe that after localisation, b is a $V(1)$ -Hurewicz image the element β_2 constructed above; certainly an examination of Ausoni's construction of b suggests this to be true. In contrast to the intricate computational methods of [Aus10], our proof of the periodicity of β_{n+1} is achieved by detecting it modulo p as multiplication by an invertible element of the Picard-graded homotopy of the $K(n+1)$ -local Moore spectrum.

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2 Generalities on the $K(n)$ -local category

2.1 Morava K and E -theories

A central object of the chromatic program is the Honda formal group law Γ_n of height n over the finite field \mathbb{F}_{p^n} . We will write K or K_n for the 2-periodic Landweber-exact cohomology theory that supports this formal group law; thus the coefficients of K are $\pi_* K = \mathbb{F}_{p^n}[u^{\pm 1}]$.

The (Lubin-Tate) universal deformation of Γ_n is defined over the ring³ $\mathbb{W}(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$. There is a cohomology theory E_n – *Morava E -theory* – whose homotopy groups are given by that ring, and support this formal group law. The theorem of Goerss-Hopkins-Miller [GH04] equips E_n with the structure of an E_∞ ring spectrum.

³Here $\mathbb{W}(\mathbb{F}_{p^n})$ is the ring of Witt vectors over \mathbb{F}_{p^n} , alternatively defined by adjoining a primitive $p^n - 1^{\text{st}}$ root of unity to the p -adic integers, \mathbb{Z}_p .

If we write \mathfrak{m} for the maximal ideal $\mathfrak{m} = (p, u_1, \dots, u_{n-1}) \subseteq \pi_*(E_n)$, these cohomology theories are related via:

$$K^*(X) \cong (E_n/\mathfrak{m})^*(X) \cong \mathbb{F}_{p^n} \otimes_{\mathbb{F}_p} K(n)^*(X)[u]/(u^{p^n-1} - v_n),$$

and similarly in homology. Notice that the representing spectrum K is equivalent to a bouquet of copies of the “standard” Morava K-theory, $K(n)$. Thus its Bousfield class is the same as that of $K(n)$.

Recall that $\mathbb{F}_{p^n}^\times$ consists of $p^n - 1$ st roots of unity. Factoring $p^n - 1 = (p - 1)(1 + p + \dots + p^{n-1})$, we note that $2(p - 1)$ divides $p^n - 1$ when p is odd and n is even. Define a $p - 1$ st root of $(-1)^{n-1}$, $\xi \in \mathbb{F}_{p^n}$, by:

$$\xi := \begin{cases} 1, & n \text{ odd} \\ \text{a primitive } 2p - 2 \text{nd root of } 1, & n \text{ even.} \end{cases}$$

2.2 The Morava stabiliser group

We will briefly summarise some preliminaries regarding the Morava stabiliser group. We refer the reader to [Mor85] and [GHMR05] for more complete discussions of this material.

We will use the notation $\mathbb{S}_n := \text{Aut}_{\mathbb{F}_{p^n}}(\Gamma_n)$ to denote the group of automorphisms of Γ_n . Then \mathbb{G}_n is defined as the semidirect product

$$\mathbb{G}_n := \mathbb{S}_n \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p).$$

The Goerss-Hopkins-Miller theorem lifts the defining action \mathbb{G}_n on Γ_n to a continuous action on the spectrum E_n through E_∞ maps. The work of Devinatz-Hopkins [DH04] and Davis [Dav06] constructs homotopy fixed point spectra with respect to closed subgroups of \mathbb{G}_n and associated descent spectral sequences.

We write \mathcal{O}_n for the noncommutative ring

$$\mathcal{O}_n = \mathbb{W}(\mathbb{F}_{p^n})\langle S \rangle / (S^n - p, Sa = a^\sigma S)$$

where σ denotes a lift of the Frobenius on \mathbb{F}_{p^n} to the ring $\mathbb{W}(\mathbb{F}_{p^n})$ of Witt vectors. Then $\mathcal{O}_n \cong \text{End}_{\mathbb{F}_{p^n}}(\Gamma_n)$ is the ring of endomorphisms of Γ_n , and so $\mathbb{S}_n \cong \mathcal{O}_n^\times$.

Now, \mathbb{S}_n naturally acts on \mathcal{O}_n (by right multiplication) through left $\mathbb{W}(\mathbb{F}_{p^n})$ -module homomorphisms, and so defines a map $\mathbb{S}_n \rightarrow \text{GL}_n(\mathbb{W}(\mathbb{F}_{p^n}))$, since \mathcal{O}_n is free of rank n over $\mathbb{W}(\mathbb{F}_{p^n})$, with basis $\{1, S, \dots, S^{n-1}\}$. It turns out that the determinant of elements coming from \mathbb{S}_n actually lie in \mathbb{Z}_p^\times , instead of $\mathbb{W}(\mathbb{F}_{p^n})^\times$; this gives a homomorphism

$$\det : \mathbb{G}_n \rightarrow \mathbb{Z}_p^\times$$

by first projecting away the $\text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ factor. We will write $S\mathbb{G}_n$ for the kernel of this homomorphism.

We recall that there is an isomorphism $\mathbb{Z}_p^\times = \mu_{p-1} \times (1 + p\mathbb{Z}_p)^\times \cong \mu_{p-1} \times \mathbb{Z}_p$. Composing the determinant with the projection onto the second factor yields the reduced determinant $\overline{\det} : \mathbb{G}_n \rightarrow \mathbb{Z}_p$. If we write \mathbb{G}_n^1 for the kernel of $\overline{\det}$, then there is a split short exact sequence

$$1 \rightarrow \mu_{p-1} \rightarrow \mathbb{G}_n^1 \rightarrow S\mathbb{G}_n \rightarrow 1$$

Furthermore, one may define a homomorphism $z : \mathbb{Z}_p^\times \rightarrow \mathbb{G}_n$ by $a \mapsto a + 0 \cdot S + \dots + 0 \cdot S^{n-1}$. The image is central, and the composite $\det \circ z : \mathbb{Z}_p^\times \rightarrow \mathbb{Z}_p^\times$ is $a \mapsto a^n$. In particular, if n is coprime to both p and $p - 1$, this map is an isomorphism, yielding a splitting $\mathbb{G}_n \cong S\mathbb{G}_n \times \mathbb{Z}_p^\times$. Furthermore, we get a splitting $\mathbb{G}_n \cong \mathbb{G}_n^1 \times \mathbb{Z}_p$.

2.3 The Picard group

We recall that the category of $K(n)$ -local spectra is symmetric monoidal; the tensor product

$$A \otimes B := L_{K(n)}(A \wedge B)$$

is the $K(n)$ -localisation of the smash product of A and B . The unit is the $K(n)$ -local sphere $S := L_{K(n)}(S^0)$.

An *invertible spectrum* A is invertible with respect to this tensor product; that is, there exists another spectrum B with $A \otimes B \simeq S$. The set of weak equivalence classes of invertible spectra forms an abelian group under tensor product, the *Picard group* of the $K(n)$ -local category, denoted Pic_n .

It was shown in [HMS94] that A is invertible if and only if $K(n)_*(A)$ is free of rank one over $K(n)_*$. Consequently there is a well-defined *dimension* $\dim(A) \in \mathbb{Z}/2(p^n - 1)$ which records the degree of a generator of $K(n)_*(A)$.

For spectra A, B , $F(A, B)$ will denote the function spectrum of maps from A to B . If A is an element of Pic_n , it is easily seen that the Spanier-Whitehead dual $F(A, S) = A^{-1}$ is the inverse to A : the evaluation map $A \otimes F(A, S) \rightarrow S$ is an equivalence. We record the following:

Proposition 2.1. *For any $K(n)$ -local spectrum E , the localisation map $A \wedge E \rightarrow A \otimes E$ is an equivalence.*

Proof. It suffices to see that the domain is local. Since A is $K(n)$ -finite, it is dualisable [HS99]. Thus $A \wedge E = F(A^{-1}, S) \wedge E = F(A^{-1}, E)$ is local, since E is. \square

2.4 Pic_n -graded invariants

In [HMS94] it was suggested that one should take a larger view of homotopy groups in the $K(n)$ -local category, and grade homotopy by the Picard group. To invent some notation, we will write $\pi_A(X) := [A, X]$, for $K(n)$ -local spectra X and invertible A ; then $\pi_{S^n}(X) = \pi_n(X)$.

We propose that the same approach be used to index the coefficients of more general (co)homology theories.

Definition 2.2. Let X and E be $K(n)$ -local spectra. For an invertible spectrum A , define

$$E_A(X) := [A, X \wedge E], \quad E_A^\vee(X) := [A, X \otimes E], \quad \text{and} \quad E^A(X) := [X, A \otimes E].$$

These are the Pic_n -graded E -homology, *completed E -homology*, and *E -cohomology*.

We note that $E_A(X)$ need not be isomorphic to the $E_A^\vee(X)$, even for $A = S^m$. Indeed, when $E = E_n$ is Morava E -theory, the completed homology theory is the *Morava module* of X , which may differ from $(E_n)_m(X)$ when X is an infinite complex. Note however, that since $A \wedge E \simeq A \otimes E$, there is no distinction between the analogues in cohomology.

To distinguish from the standard indexing (by elements $m \in \mathbb{Z}$, corresponding to S^m), we will write

$$E_\star(X) := \bigoplus_{A \in \text{Pic}_n} E_A(X), \quad \text{but} \quad E_*(X) := \bigoplus_{n \in \mathbb{Z}} E_n(X)$$

and similarly for completed homology and cohomology.

Example 2.3. Since K satisfies a strong Künneth theorem,

$$K^A(X) = [X, A \otimes K] = [X \otimes A^{-1}, K] = K^0(X \otimes A^{-1}) = (K^*(X) \otimes_{K_*} K^*(A^{-1}))_0$$

If A has dimension d , then A^{-1} has dimension $-d$, so we conclude that $K^A(X) \cong K^d(X)$. So grading K^\bullet by the full Picard group does not recover any more information than grading by the integers.

The same argument works if one replaces K with E_n , since $E_n^*(A^{-1})$ is free of rank 1 over E_n^* , giving a collapsing Künneth spectral sequence. However, the result is far from true for all (co)homology theories, most notably stable homotopy.

Additionally, we note the following, whose proof is immediate:

Proposition 2.4. *If A is invertible and E a $K(n)$ -local ring spectrum, then $E_\bullet(A) \cong E_\bullet^\vee(A)$ and $E^\bullet(A)$ are free $E_\bullet := E_\bullet(S)$ -modules of rank one.*

Note that, by the (collapsing) Künneth spectral sequence, this implies that for any spectrum X , the natural map

$$E_\bullet(A) \otimes_{E_\bullet} E_\bullet(X) \rightarrow E_\bullet(A \otimes X)$$

is an isomorphism (and similarly for the completed homology and cohomology).

2.5 Localising ring spectra

If X is a $K(n)$ -local ring spectrum with multiplication μ , $A \in \text{Pic}_n$, and $f : A \rightarrow X$ an element of $\pi_A(X)$, one can localise X away from f in the same fashion as is usually done for homotopy elements.

Definition 2.5. Define $X[f^{-1}]$ to be the telescope (homotopy colimit) of the diagram

$$X \xrightarrow{m_f} A^{-1} \otimes X \xrightarrow{1 \otimes m_f} A^{-1} \otimes (A^{-1} \otimes X) \xrightarrow{1 \otimes 1 \otimes m_f} \dots$$

Here m_f is the composite

$$X = A^{-1} \otimes A \otimes X \xrightarrow{1 \otimes f \otimes 1} A^{-1} \otimes X \otimes X \xrightarrow{1 \otimes \mu} A^{-1} \otimes X$$

where μ is the multiplication in X .

As in section V.2 of [EKMM97], if X is an E_∞ -ring spectrum, then so too is $X[f^{-1}]$. Standard properties of homotopy colimits then give:

Proposition 2.6. *The natural map $X \rightarrow X[f^{-1}]$ induces an isomorphism*

$$(K_\bullet X)[f^{-1}] \rightarrow K_\bullet(X[f^{-1}]).$$

where f is regarded as an element of $K_A(X) = [A, X \otimes K]$ by smashing with the unit of K . This induces an isomorphism

$$(K_* X)[f_*^{-1}] \rightarrow K_*(X[f^{-1}]).$$

where $f_* \in K_{\dim(A)}(X)$ is the image of a generator of $K_*(A)$ under f . The same holds with K replaced by E_n .

Write η for the unit in X . By construction, there is an equivalence $\mu \circ (f \otimes 1) : A \otimes X[f^{-1}] \rightarrow X[f^{-1}]$; let g be the inverse equivalence. Define $f^{-1} : A^{-1} \rightarrow X[f^{-1}]$ as

$$A^{-1} = A^{-1} \otimes S \xrightarrow{1 \otimes \eta} A^{-1} \otimes X[f^{-1}] \xrightarrow{1 \otimes g} A^{-1} \otimes A \otimes X[f^{-1}] = X[f^{-1}]$$

By construction $\eta = f \cdot f^{-1} : S = A \otimes A^{-1} \rightarrow X[f^{-1}]$; thus f is indeed invertible in $\pi_\bullet(X[f^{-1}])$.

2.6 Automorphism groups

For a spectrum A , we will write $\text{End}(A) := F(A, A)$ for the function spectrum of maps from A to itself. This is an associative ring spectrum under composition of functions, and so $\pi_0 \text{End}(A, A) = [A, A]$ forms a ring.

Definition 2.7. Write $\text{Aut}(A) \subseteq \Omega^\infty \text{End}(A)$ for the union of components whose image in the ring $\pi_0 \text{End}(A, A)$ is invertible.

Note that the multiplication in $\text{End}(A)$ equips $\text{Aut}(A)$ with the structure of a grouplike A_∞ monoid. We also note that if $A = S$, the adjoint of the identity on A defines an equivalence of ring spectra $\text{End}(S) \simeq S$, and so $\text{Aut}(S)$ is identified as the A_∞ monoid of units in the ring spectrum S , $\text{GL}_1(S)$:

$$\text{Aut}(S) = \text{GL}_1(S) \subseteq \Omega^\infty S$$

The following is immediate:

Proposition 2.8. *If A is invertible and B any spectrum, tensoring with the identity of A gives an equivalence*

$$id_A \otimes - : \text{End}(B) \rightarrow \text{End}(A \otimes B).$$

Passing to infinite loop spaces yields a natural equivalence of A_∞ monoids $\text{Aut}(B) \rightarrow \text{Aut}(A \otimes B)$.

In particular, taking $B = S$, we conclude

Corollary 2.9. *If A is invertible, smashing with the identity of A defines an equivalence of A_∞ monoids*

$$\text{GL}_1(S) \rightarrow \text{Aut}(A).$$

Loosely speaking, A becomes a $\text{GL}_1(S)$ -spectrum by its action on the left tensor factor of $S \otimes A = A$.

2.7 Thom spectra

Let X be a topological space, and $\zeta : X \rightarrow B\text{GL}_1(S)$ a continuous map. Ando, et al. define the Thom spectrum X^ζ in [ABG⁺08] as

$$X^\zeta := \Sigma^\infty P_+ \wedge_{\Sigma^\infty \text{GL}_1(S)_+} S$$

where P is the principle $\text{GL}_1(S)$ -bundle over X defined by ζ . Using Corollary 2.9, we may modify and extend this definition in the $K(n)$ -local category:

Definition 2.10. For an invertible spectrum A , define the *Thom spectrum* $X^{A\zeta}$ as

$$X^{A\zeta} := \Sigma^\infty P_+ \otimes_{\Sigma^\infty \text{GL}_1(S)_+} A = L_{K(n)}(\Sigma^\infty P_+ \wedge_{\Sigma^\infty \text{GL}_1(S)_+} A).$$

Note that when $A = S$, this differs slightly from the definition in [ABG⁺08] in that we have $K(n)$ -localised the Thom spectrum. This extension of the definition of Thom spectra to have any invertible spectrum as “fibre” is convenient, but not very substantial; noticing that the action of $\text{GL}_1(S)$ is on the S factor in $A = S \otimes A$, one can easily show:

Proposition 2.11. $X^{A\zeta} = X^\zeta \otimes A$.

The composite

$$X \times X \xrightarrow{pr_1} X \xrightarrow{\zeta} BGL_1(S)$$

of ζ with the projection onto the first factor defines a Thom spectrum over $X \times X$, which may be identified with $X^{A\zeta} \otimes X_+$. Furthermore, the diagonal map $\Delta : X \rightarrow X \times X$ is covered by a map of Thom spectra

$$D : X^{A\zeta} \rightarrow X^{A\zeta} \otimes X_+$$

since $pr_1 \circ \Delta = \text{id}_X$. The map D is the *Thom diagonal*, and for any $K(n)$ -local ring cohomology theory E , makes $E^\star(X^{A\zeta})$ into a $E^\star(X)$ -module.

Similarly, for each point $x \in X$, the inclusion $\{x\} \subseteq X$ is covered by a “fibre inclusion”

$$i_x : A = X^{A\zeta}|_{\{x\}} \rightarrow X^{A\zeta}.$$

Definition 2.12. A class $u \in E^A(X^{A\zeta})$ is a *Thom class* if, for every $x \in X$ its restriction along $i_x : A \rightarrow X^{A\zeta}$, $i_x^*(u) \in E^A(A) \cong E_0$ is a unit.

Proposition 2.13. *If $X^{A\zeta}$ admits a Thom class u for E , then $E^\star(X^{A\zeta})$ is a free $E^\star(X)$ -module of rank one, generated by u .*

When $A = S$, this is classical, and is realised in homology by the map:

$$E \otimes X^{A\zeta} \xrightarrow{1 \otimes D} E \otimes X^{A\zeta} \otimes X_+ \xrightarrow{1 \otimes u \otimes 1} E \otimes E \otimes A \otimes X \xrightarrow{\mu \otimes 1} E \otimes A \otimes X.$$

See, e.g., [MR81] or [ABG⁺08], Prop 5.43. For general A , this is obtained from that fact and the Pic_n -graded isomorphism

$$E^\star(X^{A\zeta}) = E^\star(X^\zeta \otimes A) \cong E^\star(X^\zeta) \otimes_{E_\star} E_\star A.$$

Let $\alpha : A \rightarrow E$ be any map, and write $X^{E\zeta}$ for the E -module Thom spectrum associated to the map $BGL_1(\eta) \circ \zeta : X \rightarrow BGL_1 E$. We will lift α to map of generalised Thom spectra:

Definition 2.14. Define $Th(\alpha) : X^{A\zeta} \rightarrow X^{E\zeta}$ as the composite

$$X^{A\zeta} = (\Sigma^\infty P_+ \otimes_{\Sigma^\infty GL_1(S)_+} S) \otimes A \xrightarrow{1 \otimes \eta \otimes \alpha} (\Sigma^\infty P_+ \otimes_{\Sigma^\infty GL_1(S)_+} E) \otimes E = X^{E\zeta} \otimes E \xrightarrow{\mu} X^{E\zeta}$$

3 Eilenberg-MacLane spaces

3.1 Recollections from Ravenel-Wilson

Working stably, p -locally, we note the equivalences⁴

$$\Sigma^\infty K(\mu_{p^\infty}, n)_+ = \varinjlim \Sigma^\infty K(\mathbb{Z}/p^j, n)_+ \simeq \Sigma^\infty K(\mathbb{Z}_p, n+1)_+ \simeq \Sigma^\infty K(\mathbb{Z}, n+1)_+ \quad (3.1.1)$$

(The first uses the Bockstein). This will be our main object of study:

Definition 3.1. Let $X = X_n$ denote the $K(n)$ -localisation of the unreduced suspension spectrum of $K(\mathbb{Z}_p, n+1)$,

$$X_n = L_{K(n)} \Sigma^\infty K(\mathbb{Z}_p, n+1)_+.$$

⁴Here $\mu_{p^\infty} \cong \varinjlim \mathbb{Z}/p^j$ is the group of p^{th} power roots of unity in, e.g., \mathbb{C} .

We recall from Ravenel-Wilson [RW80] the Morava K-theory of this spectrum:

$$K(n)^*K(\mathbb{Z}_p, n+1) = K(n)_*[[x]], \quad \text{and} \quad K(n)_*K(\mathbb{Z}_p, n+1) = \bigotimes_{k \geq 0} R(b_k).$$

Here $|x| = 2g(n)$, where we define $g(n) := \frac{p^n-1}{p-1}$. In the notation of section 12 of [RW80], x corresponds to the class x_S for $S = (1, 2, \dots, n-1)$. Also, for each integer $k \geq 0$, $R(b_k)$ is the ring

$$R(b_k) := K(n)_*[b_k]/(b_k^p - (-1)^{n-1}v_n^{p^k}b_k) = \mathbb{F}_p[x, v_n^\pm]/(b_k^p - (-1)^{n-1}v_n^{p^k}b_k),$$

where the class b_k has dimension $2p^k g(n)$ and is dual to $(-1)^{k(n-1)}x^{p^k}$. The notation b_k is our abbreviation for Ravenel-Wilson's b_J , with $J = (nk, 1, 2, \dots, n-1)$.

There are similar results for $K(\mathbb{Z}/p^j, n)$:

$$K(n)^*K(\mathbb{Z}/p^j, n) = K(n)_*[x]/x^{p^j}, \quad \text{and} \quad K(n)_*K(\mathbb{Z}/p^j, n) = \bigotimes_{k=0}^{j-1} R(b_k)$$

We have normalised these classes to be consistent across the limit in (3.1.1). This is not quite consistent with the notation of section 11 of [RW80]; there $K(n)_*K(\mathbb{Z}/p^j, n)$ is presented as being generated by classes a_I , with $I = (nk, n(j-1)+1, n(j-1)+2, \dots, n(j-1)+n-1)$ with $0 \leq k < j$. This a_I differs from b_k by a power of v_n .

In the extension

$$K^*X_n = \mathbb{F}_{p^n} \otimes_{\mathbb{F}_p} K(n)^*(X_n)[u]/(u^{p^n-1} - v_n),$$

use the $2p - 2^{\text{nd}}$ root of unity, ξ , to define a new (degree 0) coordinate $y := \xi x u^{g(n)}$; then

$$K^*(X_n) = K_*[[y]]$$

We may similarly normalise the K -homology; setting $c_k = (-1)^{k(n-1)}\xi^{-p^k}b_k u^{-p^k g(n)} = \xi^{-1}b_k u^{-p^k g(n)}$ we have

$$K_*(X_n) = K_*[c_0, c_1, \dots]/(c_k^p - c_k).$$

and $\langle c_k, y^{p^j} \rangle = \delta_k^j$.

Proposition 3.2. *The multiplication on X_n equips K^*X_n with the structure of a formal group over K_* which is isomorphic to the formal multiplicative group, \mathbb{G}_m .*

Proof. The computations described above equip K^*X_n with a coordinate y , and the associative and unital multiplication on X_n coming from the H-space structure on $K(\mathbb{Z}_p, n+1)$ defines a formal group law on $K^*X_n = K_*[[y]]$.

We may compute the p -series of this formal group law using [RW80]. The relevant fact is that the Verschiebung, well-defined⁵ up to powers of v_n , satisfies $V(x) = (-1)^{n-1}x$. Thus

$$[p](x) = FV(x) = (-1)^{n-1}v_n^{-1}x^p$$

Consequently $[p](y) = y^p$. Such a p -series is only possible for the multiplicative group. □

We note that this computation allows one to formally define the a -series $[a](y) \in K_*[[y]]$ for any element $a \in \mathbb{Z}_p$.

⁵Because $\mathbb{F}_p = K(n)_*/(v_n - 1)$ is a perfect field, while $K(n)_*$ is not, V is not naturally defined on $K(n)^*(X)$, but on its cyclically graded analogue $\overline{K(n)^*}(X) = K(n)^*(X)/(v_n - 1)$.

Definition 3.3. Write a in its p -adic expansion as $a = a_0 + a_1p + a_2p^2 + \dots$, where $0 \leq a_i < p$. Then

$$[a](y) := [a_0](y) +_F [a_1]([p](y)) +_F [a_2]([p^2](y)) +_F \dots$$

Here, $+_F$ is addition according to the formal group law on $K_*[[y]]$.

The formula gives a well-defined series, since $\deg([p^n](y)) = p^n$ grows with n .

3.2 Group actions

The p -adic integers \mathbb{Z}_p are a topologically cyclic group with generator $1 \in \mathbb{Z}_p$; that is, the subgroup generated by 1 (i.e., \mathbb{Z}) is dense in \mathbb{Z}_p . We will have occasion to write elements of \mathbb{Z}_p in multiplicative notation; then we will write h for the generator 1.

Furthermore, for $p > 2$,

$$\mathbb{Z}_p^\times \cong \mu_{p-1} \oplus (1 + p\mathbb{Z}_p)^\times,$$

and the latter factor is isomorphic to \mathbb{Z}_p . Let $\zeta = \zeta_{p-1} \in \mu_{p-1}$ be a primitive $(p-1)^{\text{st}}$ root of unity, i.e., a generator of μ_{p-1} . A generator for $(1 + p\mathbb{Z}_p)^\times$ is $1 + p$. Then \mathbb{Z}_p^\times is also topologically cyclic, with generator $g := (\zeta, 1 + p)$. We note that $g \bmod p = \zeta \in \mathbb{F}_p^\times$.

The group \mathbb{Z}_p^\times acts on \mathbb{Z}_p , and hence on $X_n = L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)_+$. For an element $a \in \mathbb{Z}_p^\times$, we will denote by ψ^a the map $\psi^a : X_n \rightarrow X_n$ given by the action of a .

Proposition 3.4. *The action of \mathbb{Z}_p^\times on $K^*(X_n)$ is via ring homomorphisms, and is determined by $\psi^a(y) = [a](y)$.*

Proof. This formula in fact holds for every $a \in \mathbb{Z}_p$. Note that this homotopy commutes, for any $m \in \mathbb{N}$:

$$\begin{array}{ccc} K(\mathbb{Z}_p, n+1) & \xrightarrow{\Delta} & K(\mathbb{Z}_p, n+1)^{\times m} \\ & \searrow \psi^m & \downarrow \text{mult} \\ & & K(\mathbb{Z}_p, n+1) \end{array}$$

where Δ is the m -fold diagonal, and mult is m -fold multiplication. The path along the upper right carries y to $[m](y)$.

Being defined by space-level maps, the action of \mathbb{Z}_p on $K^*(X_n)$ may be regarded as a *continuous* homomorphism $\mathbb{Z}_p \rightarrow \text{End}(K_*[[y]])$. Here, the topology on $\text{End}(K^*(X))$ is compact-open with respect to the natural topology on $K_*(Y)$ defined in section 11 of [HS99]. In that topology, the map $\mathbb{Z}_p \subseteq [X, X] \rightarrow \text{End}(K^*(X))$ is continuous. The argument above indicates that it agrees with the action $a \cdot y = [a](y)$ when $a \in \mathbb{N}$. Since the latter is also continuous, and \mathbb{N} is dense in \mathbb{Z}_p , these two actions must agree. \square

The group $\mu_{p-1} \cong \mathbb{F}_p^\times$ acts via multiplication on $\mathbb{Z}/p = \mathbb{F}_p$, and therefore on $K(\mathbb{Z}/p, n)$.

Proposition 3.5. *The action of μ_{p-1} on $K^*K(\mathbb{Z}/p, n) \cong K_*[y]/y^p$ is given by $\psi^\zeta(y^m) = \zeta^m y^m$.*

Proof. Since the action is space-level, it suffices to show that $\psi^\zeta(y) = \zeta y$. To see this, we note that the following diagram commutes:

$$\begin{array}{ccc} K(\mathbb{Z}/p, 1) \times K(\mathbb{Z}/p, 1)^{\times n-1} & \xrightarrow{\circ} & K(\mathbb{Z}/p, n) \\ \psi^\zeta \times 1 \downarrow & & \downarrow \psi^\zeta \\ K(\mathbb{Z}/p, 1) \times K(\mathbb{Z}/p, 1)^{\times n-1} & \xrightarrow[\circ]{} & K(\mathbb{Z}/p, n) \end{array}$$

since both paths around the diagram represent ζ times the fundamental class in $H^n(K(\mathbb{Z}/p, 1)^{\times n}; \mathbb{F}_p)$. Thus in $K(n)_*$,

$$\begin{aligned}\psi_*^\zeta(b_0) &= \psi_*^\zeta(a_{(0)} \circ a_{(1)} \circ \cdots \circ a_{(n-1)}) \\ &= \psi_*^\zeta(a_{(0)}) \circ a_{(1)} \circ \cdots \circ a_{(n-1)} \\ &= \zeta a_{(0)} \circ a_{(1)} \circ \cdots \circ a_{(n-1)} \\ &= \zeta b_0.\end{aligned}$$

The third equality uses the claim that $\psi_*^\zeta(a_0) = \zeta a_{(0)} \in K(n)_2 K(\mathbb{Z}/p, 1)$. This follows from the fact that both classes are carried injectively by the Bockstein to the unique class in $K(n)_2 K(\mathbb{Z}_p, 2)$ which is ζ times the Hurewicz image of the fundamental class of $S^2 \rightarrow K(\mathbb{Z}_p, 2)$. The claimed result then follows by duality. \square

The commutativity of the diagram of exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}_p & \xrightarrow{p} & \mathbb{Z}_p & \xrightarrow{\text{mod } p} & \mathbb{Z}/p \longrightarrow 0 \\ & & \downarrow g & & \downarrow g & & \downarrow \zeta \\ 0 & \longrightarrow & \mathbb{Z}_p & \xrightarrow{p} & \mathbb{Z}_p & \xrightarrow{\text{mod } p} & \mathbb{Z}/p \longrightarrow 0 \end{array}$$

yields a commutative diagram of fibrations

$$\begin{array}{ccccc} K(\mathbb{Z}/p, n) & \xrightarrow{\beta} & K(\mathbb{Z}_p, n+1) & \xrightarrow{\psi^p} & K(\mathbb{Z}_p, n+1) \\ \psi^\zeta \downarrow & & \downarrow \psi^g & & \downarrow \psi^g \\ K(\mathbb{Z}/p, n) & \xrightarrow{\beta} & K(\mathbb{Z}_p, n+1) & \xrightarrow{\psi^p} & K(\mathbb{Z}_p, n+1) \end{array} \quad (3.2.2)$$

Thus, the Bockstein

$$\beta : L_{K(n)} \Sigma^\infty K(\mathbb{Z}/p, n)_+ \rightarrow L_{K(n)} \Sigma^\infty K(\mathbb{Z}_p, n+1)_+ = X_n$$

is \mathbb{Z}_p^\times -equivariant where the action of \mathbb{Z}_p^\times on $L_{K(n)} \Sigma^\infty K(\mathbb{Z}/p, n)_+$ is via the reduction $\mathbb{Z}_p^\times \rightarrow \mu_{p-1}$. In particular, we conclude that

$$\psi^g(y) \bmod y^p = \zeta y$$

3.3 Splitting $K(\mathbb{Z}/p, n)$

For any p -complete spectrum Y , there is also an action (distinct from the action described in the previous section) of \mathbb{Z}_p^\times on Y where group elements acts in homotopy by multiplication; we will write the action of ζ simply as ζ .

Definition 3.6. Let $Z = Z_n$ to be the homotopy fibre of the self map

$$\psi^\zeta - \zeta : L_{K(n)} \Sigma_+^\infty K(\mathbb{Z}/p, n) \rightarrow L_{K(n)} \Sigma_+^\infty K(\mathbb{Z}/p, n)$$

Lemma 3.7. *Z is an element of Pic_n . Furthermore,*

$$L_{K(n)}\Sigma_+^\infty K(\mathbb{Z}/p, n) \simeq \bigvee_{k=0}^{p-1} Z^{\otimes k}.$$

Lastly, $Z^{\otimes p-1} \simeq S$.

Proof. By construction, $K^*(Z)$ is the subspace of $K^*K(\mathbb{Z}/p, n) = K_*[y]/y^p$ where ψ^ζ acts with eigenvalue ζ . By Proposition 3.5, this is the K_* -subspace generated by y . This has rank 1, so Z is invertible.

It is apparent that the direct sum decomposition

$$K^*K(\mathbb{Z}/p, n) = \bigoplus_{k=0}^{p-1} K_*\{y^k\}$$

is an eigenspace decomposition for the ψ^ζ (with the subspace generated by y^k having eigenvalue ζ^k). Consequently $L_{K(n)}\Sigma_+^\infty K(\mathbb{Z}/p, n)$ decomposes as a bouquet $\bigvee_k Z(\zeta^k)$, where the eigenspectrum $Z(\zeta^k)$ is the homotopy fibre of $\psi^\zeta - \zeta^k$.

Since x^k is dual to b_0^k for $k < p$, we may identify the image of $K_*Z(\zeta^k)$ in $K_*K(\mathbb{Z}/p, n)$ as the subspace generated by b_0^k . Thus the map $Z^{\otimes k} \rightarrow Z(\zeta^k)$ induced by k -ary multiplication in $K(\mathbb{Z}/p, n)$ is an isomorphism in K_* , which identifies the other summands as tensor powers of Z .

The same argument, applied to $k = p$ (along with the fact that $b_0^p = (-1)^{n-1}v_nb_0$) yields $Z^{\otimes p} \simeq Z$. The last result follows by cancelling a factor of Z . \square

3.4 An invariant description of $K_*K(\mathbb{Z}_p, n+1)$

Definition 3.8. Let G be a profinite group and let R be a topological ring which is complete with respect to an ideal \mathfrak{m} . Suppose further that R admits a continuous action of G . Define $R[[G]]$ as the *twisted, completed group ring on G* :

$$R[[G]] := \varprojlim_j \varprojlim_U R/\mathfrak{m}^j[G/U]$$

where U ranges over open subgroups of G .

Multiplication in $R[[G]]$ is given by extending the formula $(r_1g_1) \cdot (r_2g_2) = [r_1(g \cdot r_2)]g_1g_2$ bi-additively. Furthermore, $R[[G]]$ admits the structure of a left R -Hopf algebra where each $g \in G$ is grouplike: $\Delta(g) = g \otimes g$.

One may similarly define the R -Hopf algebra $C(G, R)$ to be the ring of continuous R -valued functions on G . The coproduct is the dual of multiplication in G . Theorem 5.4 of [Hov04] and the discussion that follows it give a proof that these two Hopf algebras are dual to each other over R :

$$R[[G]] \cong \text{Hom}_R(C(G, R), R)$$

When $G = \mathbb{Z}_p$ and $R = \mathbb{F}_p$, a natural family of functions is given as follows: if $m = m_0 + m_1p + m_2p^2 + \dots$, define $f_k(m) = m_k$. We note that since the codomain of f_k is \mathbb{F}_p , $f_k^p = f_k$. In fact, the f_k form a set of generators for $C(\mathbb{Z}_p, \mathbb{F}_p)$, and so

$$C(\mathbb{Z}_p, \mathbb{F}_p) = \mathbb{F}_p[f_0, f_1, f_2, \dots] / (f_k^p - f_k)$$

See, e.g., section 2 of [Rav76] or 3.3 of [Hov04]. Lastly, if k is any finite extension of \mathbb{F}_p , then the natural map $C(\mathbb{Z}_p, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k \rightarrow C(\mathbb{Z}_p, k)$ is an isomorphism (this fact is perhaps more evident in the dual Hopf algebra).

Note that \mathbb{Z}_p^\times acts on \mathbb{Z}_p , and hence on $R[[\mathbb{Z}_p]]$.

Proposition 3.9. *There exists a \mathbb{Z}_p^\times -equivariant isomorphism of Hopf algebras*

$$\phi : K_*[[\mathbb{Z}_p]] \rightarrow K^*K(\mathbb{Z}_p, n+1) \cong K_*[[y]],$$

which carries a topological generator h of \mathbb{Z}_p to $1+y$.

Proof. The ring isomorphism ϕ is a basic fact about the Iwasawa algebra $\mathbb{Z}_p[[\mathbb{Z}_p]]$ (or, in this case, its reduction modulo p , and extension over \mathbb{F}_{p^n}). The dual map ϕ^* satisfies

$$\phi^*(c_k)(h^m) = \langle c_k, (1+y)^m \rangle = \binom{m}{p^k} \langle c_k, y^{p^k} \rangle = \binom{m}{p^k}$$

If we write m in its p -adic expansion as $m = m_0 + m_1p + m_2p^2 + \dots$, then by Lucas' theorem, $\binom{m}{p^k} \bmod p = m_k$. Therefore the map ϕ^* carries c_k to $f_k \in C(\mathbb{Z}_p, L_*)$. It is evidently a ring isomorphism, and so ϕ is an isomorphism of Hopf algebras.

To see that the action of \mathbb{Z}_p^\times is as claimed, we first note that \mathbb{Z}_p^\times acts on \mathbb{Z}_p through group homomorphisms; hence it acts on $K_*[[\mathbb{Z}_p]]$ through ring homomorphisms. Thus it suffices to show that for $\gamma \in \mathbb{Z}_p^\times$, $\phi(\gamma \cdot h) = \gamma \cdot \phi(h)$. The lefthand side is $\phi(h^\gamma) = \phi(h)^\gamma = (1+y)^\gamma$, whereas the right is $\gamma \cdot (1+y) = 1 + [\gamma](y)$. That these two are equal follows from the fact that y is a coordinate on the formal multiplicative group. □

3.5 Inverting roots of unity; the spectrum R_n

Define $\alpha = \beta \circ i : Z \rightarrow X$ as the composite of the Bockstein with the natural map $i : Z \rightarrow \Sigma^\infty K(\mathbb{Z}/p, n)_+$. Consider the localisation of X at this element:

Definition 3.10. Write R_n for the E_∞ ring spectrum $R_n := X[\alpha^{-1}]$.

Corollary 3.11. *The dual isomorphism of Proposition 3.9 localises to a ring isomorphism*

$$\phi^* : K_*(R_n) \rightarrow C(\mathbb{Z}_p^\times, K_*)$$

Proof. Proposition 2.6 implies that

$$K_*(X[\alpha^{-1}]) = K_*(X)[\alpha_*^{-1}] = C(\mathbb{Z}_p, K_*)[f_0^{-1}]$$

since the image of a generator of $K_*(Z)$ under α is $b_0 = \xi f_0 u^{g(n)}$. Note that $m \in \mathbb{Z}_p$ lies in \mathbb{Z}_p^\times if and only if $f_0(m) = m \bmod p$ is invertible. Therefore $C(\mathbb{Z}_p, K_*)[f_0^{-1}]$ may be identified as $C(\mathbb{Z}_p^\times, K_*)$. □

We would like a \mathbb{Z}_p^\times -equivariant version of this result. However, it is not immediately obvious that the action of \mathbb{Z}_p^\times on X localises to an action on $X[\alpha^{-1}]$, as we have not been very careful with our point-set level construction of the localisation. Happily, we may avoid this issue⁶ by simply

⁶In section 3.8, we will show that $X[\alpha^{-1}]$ is homotopy equivalent to the spectrum $E_n^{hS\mathbb{G}_n}$ which has an evident action of \mathbb{Z}_p^\times that rectifies the action that we are failing to construct here.

constructing a map which a topological generator g of \mathbb{Z}_p^\times “should” act by, and verify that induces the expected action in K_* .

Diagram (3.2.2) naturally extends to the left along $i : Z \rightarrow \Sigma^\infty K(\mathbb{Z}/p, n)_+$ to give

$$\psi^g \circ \alpha \simeq \alpha \circ \zeta : Z \rightarrow X$$

since $\zeta \simeq \psi^\zeta$ on Z . Along with the fact that ψ^g acts on X through ring spectrum maps, we get a homotopy commuting diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{=} & Z^{-1} \otimes Z \otimes X & \xrightarrow{1 \otimes \alpha \otimes 1} & Z^{-1} \otimes X \otimes X & \xrightarrow{1 \otimes \mu} & Z^{-1} \otimes X \\ \psi^g \downarrow & & \downarrow \zeta^{-1} \otimes \zeta \otimes \psi^g & & \downarrow \zeta^{-1} \otimes \psi^g \otimes \psi^g & & \downarrow \zeta^{-1} \otimes \psi^g \\ X & \xrightarrow{=} & Z^{-1} \otimes Z \otimes X & \xrightarrow{1 \otimes \alpha \otimes 1} & Z^{-1} \otimes X \otimes X & \xrightarrow{1 \otimes \mu} & Z^{-1} \otimes X \end{array}$$

In short, $m_\alpha \circ \psi^g \simeq (\zeta^{-1} \otimes \psi^g) \circ m_\alpha$.

Therefore, this diagram commutes:

$$\begin{array}{ccccccc} X & \xrightarrow{m_\alpha} & Z^{-1} \otimes X & \xrightarrow{1 \otimes m_\alpha} & Z^{-1} \otimes (Z^{-1} \otimes X) & \xrightarrow{1 \otimes 1 \otimes m_\alpha} & \dots \\ \psi^g \downarrow & & \downarrow \zeta^{-1} \otimes \psi^g & & \downarrow \zeta^{-1} \otimes (\zeta^{-1} \otimes \psi^g) & & \\ X & \xrightarrow{m_\alpha} & Z^{-1} \otimes X & \xrightarrow{1 \otimes m_\alpha} & Z^{-1} \otimes (Z^{-1} \otimes X) & \xrightarrow{1 \otimes 1 \otimes m_\alpha} & \dots \end{array}$$

Passing to homotopy colimits, we get a well-defined map $\psi^g : X[\alpha^{-1}] \rightarrow X[\alpha^{-1}]$. Since this commutes with the natural map $X \rightarrow X[\alpha^{-1}]$, we see that the action of ψ^g on $K_*X[\alpha^{-1}]$ is the localisation of the action of ψ^g on K_*X . Summarising, we have:

Lemma 3.12. *There is a self map $\psi^g : X[\alpha^{-1}] \rightarrow X[\alpha^{-1}]$ whose induced map in $K_*X[\alpha^{-1}] = C(\mathbb{Z}_p^\times, K_*)$ can be identified with translation of functions by g .*

3.6 Invertible spectra as homotopy fibres

Proposition 3.13. *Let $\gamma \in \mathbb{Z}_p^\times$. The homotopy fiber F_γ of $(\psi^g - \gamma) : R_n \rightarrow R_n$ is an invertible spectrum. When $\gamma = 1$, F_1 is equivalent to S .*

Proof. As in Proposition 3 of [Lur10], it is straightforward to see that $\psi^g - \gamma$ is surjective in K_* , with kernel consisting of those functions $f : \mathbb{Z}_p^\times \rightarrow K_*$ which satisfy $f(gx) = \gamma f(x)$. As such a function is determined by its value on 1, this has rank one over K_* . Thus $K_*(F_\gamma) = \ker((\psi^g - \gamma)_*)$ is rank one over K_* , and hence invertible.

When $\gamma = 1$, the kernel consists of those functions on \mathbb{Z}_p^\times which are invariant under translation by g , namely the constant functions. As in Lemma 2.5 of [HMS94], the unit of the ring spectrum R_n – induced by the basepoint inclusion in $K(\mathbb{Z}_p, n+1)$ – lifts to F_1 , and carries $K_*(S)$ onto these functions, yielding the desired isomorphism in K_* . \square

One should think of the second statement in this Proposition as saying that S is the homotopy fixed point spectrum for an action of \mathbb{Z}_p^\times on R_n that we have been too lazy to actually construct.

Definition 3.14. Denote by G the invertible spectrum F_g .

Proposition 3.15. *The assignment $\gamma \mapsto F_\gamma$ defines a homomorphism $\mathbb{Z}_p^\times \rightarrow \text{Pic}_n$.*

Proof. We must show that $F_{\gamma\sigma} \simeq F_\gamma \otimes F_\sigma$. Now ψ^g is a map of ring spectra, so this diagram homotopy commutes:

$$\begin{array}{ccc} F_\gamma \otimes F_\sigma & \xrightarrow{\gamma \otimes \sigma} & F_\gamma \otimes F_\sigma \\ \downarrow i & & \downarrow i \\ R_n \otimes R_n & \xrightarrow{\psi^g \otimes \psi^g} & R_n \otimes R_n \\ \downarrow \mu & & \downarrow \mu \\ R_n & \xrightarrow{\psi^g} & R_n \end{array}$$

Therefore $\mu \circ i$ lifts to $F_{\gamma\sigma}$; this map is easily seen to be an isomorphism in K_* . \square

As we will see in later sections, the image of g in Pic_n (i.e., G) is not torsion. Consequently this homomorphism is an injection. Additionally, we will show that G is *not* a sphere unless $n = 1$.

Proposition 3.16. *The canonical map $\delta : G \rightarrow X[\alpha^{-1}]$ from the homotopy fibre $G = F_g$ lifts to a map $\rho : G \rightarrow X$:*

$$\begin{array}{ccccc} & & X & \xrightarrow{\psi^g - g} & X \\ & \nearrow \rho & \downarrow j & & \downarrow j \\ G & \xrightarrow{\delta} & X[\alpha^{-1}] & \xrightarrow{\psi^g - g} & X[\alpha^{-1}] \end{array}$$

Furthermore, there are equivalences of ring spectra

$$X[\rho^{-1}] \xrightarrow{\simeq} X[\alpha^{-1}][\delta^{-1}] \xleftarrow{\simeq} X[\alpha^{-1}].$$

Here j is the natural map from a ring spectrum to its localisation.

Proof. We note that since G is invertible it is a compact object, and so the map δ must lift to one of the terms in the telescope. That is, there must then be a map $\rho : G \rightarrow Z^{-n} \otimes X$ which lifts δ . Composing with m_α an appropriate number of times if necessary, we may take n to be a multiple of $p - 1$, and get the desired map $\rho : G \rightarrow X$.

The right equivalence is the localisation map at δ . By the proof of Proposition 3.13, the image of a generator for G under δ is $f_0(x) = x \bmod p$. As this is clearly invertible in $C(\mathbb{Z}_p^\times, K_*) = K_*(X[\alpha^{-1}])$, m_δ is an equivalence, and thus the directed system defining $X[\alpha^{-1}][\delta^{-1}]$ is constant.

The left equivalence is induced by the map $X \rightarrow X[\alpha^{-1}]$ after localisation at ρ (which maps to localisation at δ , since $j\rho = \delta$). By the same argument as above, the image of a generator for G under ρ is a function $f : \mathbb{Z}_p \rightarrow K_*$ which restricts to f_0 along the inclusion $\mathbb{Z}_p^\times \subseteq \mathbb{Z}_p$. Since the former is dense in the latter, there is a unique such function f , which must be $f(x) = f_0(x) = x \bmod p$. Thus the localisation map $X[\rho^{-1}] \rightarrow X[\alpha^{-1}][\delta^{-1}]$ is an K_* -isomorphism. \square

We note that since $\text{im}(\rho_*)$ is generated by the class⁷ f_0 , we may conclude that $\dim(G) = \dim(f_0) = 2g(n)$.

⁷This class is detected by the primitive element $y \in K^*X$. We thank Mike Hopkins and Jacob Lurie for pointing out that this observation may be promoted to the claim that G can be constructed as $\Sigma \text{Cotor}_X(S, S)$; here X is a coalgebra spectrum, and $\text{Cotor}_X(S, S)$ is the associated (reduced) cobar construction.

3.7 Splitting R_n

There is an analogue for R_n of the splitting of p -adic K -theory into a wedge of $p - 1$ Adams summands:

Proposition 3.17. *There is an equivalence*

$$\bigvee_{k=0}^{p-2} G^{\otimes k} \otimes R_n^{h\mu_{p-1}} \rightarrow R_n.$$

Proof. There is a natural forgetful map $R_n^{h\mu_{p-1}} \rightarrow R_n$. Additionally, one may produce a map $\bigvee_{k=0}^{p-2} G^{\otimes k} \rightarrow R_n$ by wedging together powers of δ ; the product of these maps gives the desired equivalence.

To see that this map is an isomorphism in K_* (and hence an equivalence), we note that since

$$K_*(R_n) = C(\mathbb{Z}_p^\times, \mathbb{F}_{p^n}) = \mathbb{F}_{p^n}[f_0, f_1, f_2, \dots] / (f_0^{p-1} - 1, f_k^p - f_k),$$

then $K_*R_n^{h\mu_{p-1}}$ is the subspace generated by monomials in the f_i whose total degree is a multiple of $p - 1$, since $\psi^\zeta(f_k) = \zeta f_k$. The whole space is a free module over this subalgebra, generated by the classes $\{1, f_0, f_0^2, \dots, f_0^{p-2}\}$, which are the images of $G^{\otimes k}$ under δ^k . \square

3.8 R_n as a homotopy fixed point spectrum

Proposition 3.18. *There are ring isomorphisms*

$$C(\mathbb{Z}_p, E_{n*}) \rightarrow E_{n*}^\vee(X) \quad \text{and} \quad C(\mathbb{Z}_p^\times, E_{n*}) \rightarrow E_{n*}^\vee(R_n)$$

In this description, the action of \mathbb{G}_n on both of these Morava modules is induced by the determinant homomorphism $\det : \mathbb{G}_n \rightarrow \mathbb{Z}_p^\times$ and the natural action of \mathbb{Z}_p^\times on \mathbb{Z}_p and \mathbb{Z}_p^\times .

Proof. The proof of the first fact is nearly identical to the one given in [Hov04], section 2. The action of the Adams operations ψ^k on X gives a continuous, multiplicative map

$$a : \mathbb{Z}_p \rightarrow E_n^*(X) \quad \text{by} \quad k \mapsto \psi^k(1 + \overline{y})$$

where \overline{y} is any class that reduces modulo \mathfrak{m} to $y \in K^*(X)$. This extends linearly over E_{n*} to give a continuous ring homomorphism $E_{n*}[\mathbb{Z}_p] \rightarrow E_n^*(X)$. Applying the functor $\text{Hom}_{E_{n*}}^c(-, E_{n*})$ of continuous E_{n*} -module homomorphisms into E_{n*} yields a map

$$a^* : \text{Hom}_{E_{n*}}^c(E_n^*(X), E_{n*}) \rightarrow \text{Hom}_{E_{n*}}^c(E_{n*}[\mathbb{Z}_p], E_{n*}) \cong C(\mathbb{Z}_p, E_{n*}).$$

The identification of the codomain uses the fact that $E_{n*}[\mathbb{Z}_p]$ is free. Similarly, the domain is

$$\text{Hom}_{E_{n*}}^c(E_n^*(X), \varprojlim_I (E_n/I)_*) = \varprojlim_I \text{Hom}((E_n/I)^*(X), (E_n/I)_*) = \varprojlim_I (E_n/I)_*(X) = E_{n*}^\vee(X)$$

where the limit is taken over the ideals $I = (p^{i_0}, u_1^{i_1}, \dots, u_{n-1}^{i_{n-1}})$. We know that $E_{n*}^\vee(X)$ is pro-free, concentrated in even dimensions, with reduction modulo \mathfrak{m} isomorphic to $K_*(X) = C(\mathbb{Z}_p, \mathbb{F}_{p^n}[u^{\pm 1}])$. The ring $C(\mathbb{Z}_p, E_{n*})$ is also pro-free, and $a^* : E_{n*}^\vee(X) \rightarrow C(\mathbb{Z}_p, E_{n*})$ reduces modulo \mathfrak{m} to the isomorphism $(E_n/\mathfrak{m})_*(X) \rightarrow C(\mathbb{Z}_p, \mathbb{F}_{p^n}[u^{\pm 1}])$. The same argument works for R_n in place of X .

To see that the \mathbb{G}_n action is as claimed, we employ Peterson's adaptation [Pet11] of Ravenel-Wilson's results [RW80] to identify the formal spectrum $\mathrm{Spf} E_n^* K(\mathbb{Z}_p, n+1)$ in terms of the n^{th} exterior power of the p -divisible group associated to E_n . Concretely, we recall that the action of \mathbb{S}_n on $E_{n*}(K(\mathbb{Z}_p, 2)) \cong E_{n*}(K(\mu_{p^\infty}, 1))$ is via the defining action of \mathcal{O}_n on Γ_n . Furthermore the Hopf ring circle product satisfies $a \circ b = -b \circ a$. Therefore, the action of \mathbb{S}_n on classes in $E_{n*}(K(\mathbb{Z}_p, n+1)) \cong E_{n*}(K(\mu_{p^\infty}, n))$ lying in the image of the iterated circle product

$$E_{n*}(K(\mu_{p^\infty}, 1)^{\times n}) \rightarrow E_{n*}(K(\mu_{p^\infty}, n))$$

is via the n^{th} exterior power of the defining action. All classes in $K_*(K(\mu_{p^\infty}, n))$ may be obtained this way, and the above analysis lifts this statement to E_{n*} . Therefore the action of \mathbb{S}_n (and hence⁸ \mathbb{G}_n) must be via \det . □

A corollary of this fact is that there is an E_{n*} -Hopf algebra isomorphism $E_n^*(X) \cong E_{n*}[[\mathbb{Z}_p]]$. R_n is not obviously a coalgebra spectrum, so a similar statement cannot be made. However, one can conclude that $E_n^*(R_n) \cong E_{n*}[[\mathbb{Z}_p^\times]]$ as an $E_{n*}[[\mathbb{G}_n]]$ -module.

Theorem 3.19. *The space of E_∞ maps $R_n \rightarrow E_n$ is an infinite loop space with contractible components, the set of which are in isomorphic to \mathbb{Z}_p^\times . Furthermore, the action of $\mathbb{G}_n \simeq \mathrm{Aut}_{E_\infty}(E_n)$ on this set is via the homomorphism $\det : \mathbb{G}_n \rightarrow \mathbb{Z}_p^\times$.*

Proof. The proof is essentially the same as the main technical result of [SW11], so we will be brief. The Goerss-Hopkins-Miller obstruction machinery shows that the higher homotopy groups of $\mathrm{Map}_{E_\infty}(R_n, E_n)$ vanish if the cotangent complex for the map $\mathbb{F}_{p^n} \rightarrow E_{n*}(R_n)/\mathfrak{m}$ is contractible. But the latter is

$$E_{n*}(R_n)/\mathfrak{m} \cong C(\mathbb{Z}_p^\times, \mathbb{F}_{p^n}) \cong \mathbb{F}_{p^n}[f_0, f_1, f_2, \dots]/(f_0^{p-1} - 1, f_k^p - f_k).$$

The cotangent complex is contractible because the Frobenius on this ring is evidently an isomorphism.

The set of components of $\mathrm{Map}_{E_\infty}(R_n, E_n)$ is then the set of continuous E_{n*} -algebra homomorphisms $\mathrm{Hom}_{E_{n*}\text{-alg}}^c(E_{n*}R_n, E_{n*})$, which may be identified with $\mathbb{Z}_p^\times = \mathbb{G}_n/S\mathbb{G}_n$ by the previous result. □

Corollary 3.20. *A topological generator $\varphi \in \pi_0 \mathrm{Map}_{E_\infty}(R_n, E_n)$ lifts to an E_∞ equivalence*

$$\varphi : R_n \rightarrow E_n^{hS\mathbb{G}_n}.$$

Proof. By the previous result, the action of $S\mathbb{G}_n$ on $\mathrm{Map}_{E_\infty}(R_n, E_n)$ is homotopically trivial, and so

$$\begin{aligned} \mathrm{Map}_{E_\infty}(R_n, E_n^{hS\mathbb{G}_n}) &= \mathrm{Map}_{E_\infty}(R_n, E_n)^{hS\mathbb{G}_n} \\ &= \mathrm{Map}(BS\mathbb{G}_n, \mathrm{Map}_{E_\infty}(R_n, E_n)) \\ &\simeq \mathrm{Map}(BS\mathbb{G}_n, \mathbb{Z}_p^\times) \\ &= \mathbb{Z}_p^\times \end{aligned}$$

⁸The action of the Galois group is diagonal on tensor products, and thus via the n^{th} power (that is, trivial) on circle products.

Then φ_* is an isomorphism in E_n^* since

$$E_n^*(E_n^{hS\mathbb{G}_n}) = E_{n*}[[\mathbb{G}_n/S\mathbb{G}_n]] = E_{n*}[[\mathbb{Z}_p^\times]].$$

□

Proposition 3.21. *The cohomology class $1 + y = \varphi \bmod \mathfrak{m} \in K^*(X) = [X, K]$ extends over R_n to give a map of A_∞ -ring spectra $t : R_n \rightarrow K$.*

Proof. It was shown in [SW11] that the homotopy class $1 + y$ contains an A_∞ representative⁹ using Hochschild cohomology methods. Thus it suffices to show that $(1 + y) \circ \alpha : Z \rightarrow K$ is invertible. This follows, since the image of the fundamental class of Z under α is b_0 , and $\langle 1 + y, b_0 \rangle = \xi$.

□

3.9 Gross-Hopkins duality

The previous sections imply that there is a commutative diagram

$$\begin{array}{ccc} R_n & \xrightarrow{\psi^g - g} & R_n \\ \simeq \downarrow & & \downarrow \simeq \\ E_n^{hS\mathbb{G}_n} & \xrightarrow{\psi^g - g} & E_n^{hS\mathbb{G}_n} \end{array}$$

where the action of $\psi^g \in \mathbb{Z}_p^\times$ on $E_n^{hS\mathbb{G}_n}$ is from the residual $\mathbb{G}_n/S\mathbb{G}_n$ Morava stabiliser action. Consequently the homotopy fibres of each row are equivalent. In [GHMR12], the lower homotopy fibre was named $S\langle \det \rangle$, and its Morava module was shown to be $E_{n*}^\vee(S\langle \det \rangle) = E_{n*}\langle \det \rangle$.

In [HG94] it was shown that this is the same Morava module as for $\Sigma^{n-n^2}I_n$, where I_n is the Brown-Comenetz dual of $M_n S^0$, the n^{th} monochromatic layer of the sphere spectrum. It was also shown that for $2p-2 \geq \max\{n^2, 2n+2\}$, an invertible spectrum is determined by its Morava module. We conclude:

Corollary 3.22. *There is an equivalence $G \simeq S\langle \det \rangle$. When $2p-2 \geq \max\{n^2, 2n+2\}$, these may also be identified with $\Sigma^{n-n^2}I_n$.*

4 Thom spectra and characteristic classes

4.1 An analogue of MU

Recall that for an E_∞ ring spectrum R , $\text{gl}_1 R$ is the spectrum whose infinite loop space is $\text{GL}_1 R$, the space of units in $\Omega^\infty R$. Further, gl_1 is functorial for maps of E_∞ -ring spectra.

Let $\eta : S \rightarrow R_n$ be the unit of R_n , and define C to be the cofibre of $\text{gl}_1 \eta : \text{gl}_1 S \rightarrow \text{gl}_1 R_n$. Now, ψ^g is a ring map, so there is a well defined map $\text{gl}_1 \psi^g : \text{gl}_1 R_n \rightarrow \text{gl}_1 R_n$. Since $(\psi^g - 1) \circ \eta = 0$, we see that $(\text{gl}_1(\psi^g) - 1) \circ \text{gl}_1(\eta) = 0$, and so there is a natural map $\kappa : C \rightarrow \text{gl}_1 R_n$ making this

⁹Strictly speaking, we showed this for $p = 2$, with $K(n)$ in place of K_n . The same methods apply here.

diagram commute

$$\begin{array}{ccccc}
\mathrm{gl}_1 S & \xrightarrow{\mathrm{gl}_1 \eta} & \mathrm{gl}_1 R_n & \xrightarrow{\quad} & C \xrightarrow{\gamma} \Sigma \mathrm{gl}_1 S \\
& & & \searrow \mathrm{gl}_1(\psi^g)-1 & \downarrow \kappa \\
& & & & \mathrm{gl}_1 R_n
\end{array}$$

Write i for the natural inclusion $i : \mathrm{GL}_1 A \rightarrow \Omega^\infty A$. Taking infinite loop spaces above, and employing i , we have a commuting diagram

$$\begin{array}{ccccccc}
\mathrm{GL}_1 S & \xrightarrow{\mathrm{GL}_1 \eta} & \mathrm{GL}_1 R_n & \xrightarrow{\quad} & \Omega^\infty C & \xrightarrow{\gamma} & B \mathrm{GL}_1 S \\
\downarrow i & & \downarrow i & & \downarrow \kappa & & \\
& & & \searrow \mathrm{GL}_1(\psi^g)-1 & \mathrm{GL}_1 R_n & & \\
& & & & \downarrow i & & \\
\Omega^\infty S & \xrightarrow{\eta} & \Omega^\infty R_n & \xrightarrow{\psi^g-1} & \Omega^\infty R_n & &
\end{array}$$

The horizontal maps are fibre sequences, and the vertical maps i are weak equivalences on connected components; thus by the five lemma we have:

Proposition 4.1. *$i \circ \kappa$ lifts to an equivalence of simply-connected covers*

$$\Omega^\infty C \langle 1 \rangle \simeq \Omega^\infty R_n \langle 1 \rangle$$

Definition 4.2. Let $BX_n := \Omega^\infty C \langle 1 \rangle \simeq \mathrm{GL}_1 R_n \langle 1 \rangle \simeq \Omega^\infty R_n \langle 1 \rangle$, and define MX_n as the Thom spectrum

$$MX_n := L_{K(n)} BX_n^\gamma$$

associated to the map $\gamma : BX_n \rightarrow B \mathrm{GL}_1(S)$.

Remark 4.3. This notation is meant to evoke thoughts of the complex cobordism Thom spectrum $MU = BU^\gamma$ (where γ is the tautological bundle over BU). Indeed, this is precisely the case when $n = 1$ (at least $K(1)$ -locally), since

$$R_1 = X_1[\rho^{-1}] = L_{K(1)} \Sigma^\infty \mathbb{C}P_+^\infty[\beta^{-1}] \simeq L_{K(1)} K \simeq L_{H\mathbb{Z}/p} K,$$

by Snaith's theorem. Thus BX_1 is the p -completion of BU , and so $MX_1 = L_{K(1)} MU$.

Additionally, we point out that γ is an infinite loop map when we equip BX_n with the infinite loop space structure¹⁰ $BX_n = \mathrm{GL}_1 R_n \langle 1 \rangle$. Using the methods of [LMSM86], we conclude:

Proposition 4.4. *The Thom spectrum MX_n is an E_∞ -ring spectrum.*

We notice that the map $B \mathrm{GL}_1(\eta) \circ \gamma : BX_n \rightarrow B \mathrm{GL}_1 R_n$ is definitionally nullhomotopic when we regard BX_n as $\Omega^\infty C \langle 1 \rangle$, and so the R_n -module Thom spectrum $BX_n^{R_n \gamma} \simeq (BX_n)_+ \otimes R_n$ is split. We use some of the notation of section 2.7 in the following:

¹⁰When $n = 1$, this is the multiplicative structure BU_\otimes which Thomifies to the standard multiplicative structure on MU .

Definition 4.5. The composite map

$$MX_n = BX_n \xrightarrow{\gamma} (BX_n)_+ \otimes R_n \xrightarrow{proj \otimes 1} R_n$$

(where $proj : (BX_n)_+ \rightarrow pt_+ = S^0$ is the projection) defines a class $u \in R_n^0(MX)$ which we will refer to as the *tautological R_n -Thom class for MX* .

We note that this element satisfies the usual properties of Thom classes, by construction:

Proposition 4.6. *If $i_\eta : S \rightarrow MX$ denotes the inclusion of the fibre over any point in BX_n , then $i_\eta^*(u) = \eta = 1 \in \pi_0(R_n)$ is invertible.*

Consequently, Proposition 2.13 gives a Thom isomorphism

$$R_n^\star(MX) \cong R_n^\star BX$$

of $R_n^\star BX$ -modules.

4.2 Cannibalistic classes for R_n

Consider a space Y and a based map $f : Y \rightarrow \Omega^\infty C$, or equivalently $f \in [\Sigma^\infty Y, C] = \tilde{C}^0(Y)$. One may define a Thom spectrum Y^ζ associated to the map $\zeta = \gamma \circ f$. Furthermore, there is a tautological nullhomotopy of $BGL_1(\eta) \circ \zeta = BGL_1(\eta) \circ \gamma \circ f$, and so a trivialisation of the Thom spectrum $Y^{R_n \zeta} \simeq Y_+ \otimes R_n$. One may directly obtain a Thom class u as the composite

$$Y^\zeta \xrightarrow{Th(\eta \circ f)} Y_+ \otimes R_n \xrightarrow{proj \otimes 1} R_n$$

Let $A \in \text{Pic}_n$; in this section we consider the interaction between the Thom isomorphism for $Y^{A\zeta}$ and the operations $\psi^k : R_n \rightarrow R_n$.

We note that a Thom class for $A\zeta$ is given by

$$1 \otimes u : A \otimes Y^\zeta = Y^{A\zeta} \rightarrow A \otimes R_n$$

If we insist that A is a power of G , we may make employ the periodicity of R_n to make the following definition:

Definition 4.7. The *normalised* Thom class $\bar{u} = \bar{u}(G^m \zeta) := \delta^m u \in R^0(Y)$ is defined as the composite

$$Y^{G^m \zeta} = G^m \otimes Y^\zeta \xrightarrow{1 \otimes u} G^m \otimes R_n \xrightarrow{\delta^m \otimes 1} R_n \otimes R_n \xrightarrow{\mu} R_n$$

Then for any $k \in \mathbb{Z}_p^\times$, define $\theta_k(G^m \zeta) \in R_n^0(Y)$ by the equation

$$\psi^k(\bar{u}) = \theta_k(G^m \zeta) \cdot \bar{u}.$$

Following Adams, we call $\theta_k(G^m \zeta)$ the (Bott) *cannibalistic class* of $G^m \zeta$.

Proposition 4.8. *Let ζ and ξ be Thom spectra associated to classes $f, h : Y \rightarrow \Omega^\infty C$.*

1. $\theta_k(0) = 1$.
2. $\theta_k(\zeta + \xi) = \theta_k(\zeta) \cdot \theta_k(\xi)$.
3. $\theta_g(G^m \zeta) = g^m \theta_g(\zeta)$.

The proof that θ_k is exponential depends upon the fact that a product of Thom classes for ζ and ξ is a Thom class for $\zeta + \xi$. Note that this implies that $\theta_k(\zeta)^{-1}$ exists, and is equal to $\theta_k(-\zeta)$. The second property uses the fact that $\psi^g(\delta^m \otimes u) = g^m \delta^m \otimes \psi^g(u)$.

4.3 Restricting to $K(\mathbb{Z}_p, n+1)$

We will denote by j the natural localisation map $j : X \rightarrow R_n = X[\rho^{-1}]$. This is closely related to a map defining a Thom spectrum over $K(\mathbb{Z}_p, n+1)$:

Proposition 4.9. *There is a map $e : K(\mathbb{Z}_p, n+1) \rightarrow BX_n = \Omega^\infty C\langle 1 \rangle$ whose adjoint lifts the composite*

$$\Sigma^\infty K(\mathbb{Z}_p, n+1)_+ \longrightarrow L_{K(n)} \Sigma^\infty K(\mathbb{Z}_p, n+1)_+ = X_n \xrightarrow{j^{-1}} X_n[\rho^{-1}] = R_n$$

to $R_n\langle 1 \rangle$.

Here, for a space Y and a ring cohomology theory E , 1 denotes the cohomology class $\Sigma^\infty Y_+ \rightarrow E$ given by the projection of Y_+ onto S^0 and the unit in E .

Proof. We first note that the adjoint of the composite displayed above carries $K(\mathbb{Z}_p, n+1)$ into the basepoint component $\Omega_0^\infty R_n$ of $\Omega^\infty R_n$ since its restriction to the basepoint $*$ in $K(\mathbb{Z}_p, n+1)$ is $j(*) - 1 = 1 - 1 = 0$. That is, the shift by -1 is necessary to ensure that the map is based. Then e lifts to $BX_n = \Omega_0^\infty R_n\langle 1 \rangle$ since $K(\mathbb{Z}_p, n+1)$ is simply connected. \square

Pulling γ back over e allows us to define a Thom spectrum $K(\mathbb{Z}_p, n+1)^{\gamma \circ e}$ which, for brevity, we will write as X^γ . By restriction of the Thom class u for MX along e , we may similarly conclude:

Corollary 4.10. *Multiplication by the Thom class $e^*(u)$ defines an isomorphism of $R_n^\star X$ -modules*

$$R_n^\star X = R_n^\star X \rightarrow R_n^\star(X^\gamma).$$

We note that the results of the previous section and a similar basepoint shift define, for any element $f \in R_n^0(Y)$ of a simply connected space Y , an associated Thom spectrum $Y^{f \circ \gamma}$ over Y and functorial Thom class $u \in R_n^0(Y^{f \circ \gamma})$.

Lemma 4.11. $(j-1) \cdot \theta_g(G\gamma) = \psi^g(j-1)$.

We would like to conclude that $\theta_g(G\gamma) = \frac{\psi^g(j-1)}{j-1}$, but it is not clear that the latter gives a meaningful expression in $R_n^0(X)$. This equation holds if one replaces $g = \zeta \cdot (1+p)$ with $1+p$, as

$$\frac{\psi^{1+p}(j-1)}{j-1} = 1 + j + \cdots + j^p.$$

Here we use the fact (a consequence of proof of Proposition 3.4) that $\psi^{1+p}(j) = j^{1+p}$.

Proof. Recall that the Thom class $e^*(u)$ for $G\gamma$ is $\delta \cdot (proj \otimes 1) \circ Th(\eta) = (proj \otimes 1) \circ Th(\delta)$. We note that since $\delta = (j-1) \circ \rho$, the following diagram commutes, and gives an alternate construction of $e^*(u)$:

$$\begin{array}{ccccc} & & X^{X^\gamma} & & \\ & \nearrow Th(\rho) & \downarrow Th(j-1) & & \\ X^{G\gamma} & \xrightarrow{Th(\delta)} & X \otimes R_n & \xrightarrow{proj \otimes 1} & R_n \end{array}$$

Here, X^{X^γ} is the Thom spectrum for the map

$$K(\mathbb{Z}_p, n+1) \xrightarrow{e} BX_n \xrightarrow{\gamma} BGL_1(S) \xrightarrow{\eta_X} BGL_1(X)$$

and $Th(\rho) : X^{G^\gamma} \rightarrow X^{X^\gamma}$ is the map on Thom spectra associated to $\rho : G \rightarrow X$.

Therefore, we compute:

$$\begin{aligned} (j-1) \cdot \psi^g(e^*(u)) &= (j-1) \cdot [\psi^g \circ (proj \otimes 1) \circ Th(\delta)] \\ &= (j-1) \cdot [\psi^g \circ (proj \otimes 1) \circ Th(j-1) \circ Th(\rho)] \\ &= (j-1) \cdot [(proj \otimes 1) \circ Th(\psi^g(j-1)) \circ Th(\rho)] \\ &= \psi^g(j-1) \cdot (proj \otimes 1) \circ Th(j-1) \circ Th(\rho) \\ &= \psi^g(j-1) \cdot e^*(u) \end{aligned}$$

□

4.4 A zero section map

Definition 4.12. For an R_n -oriented Thom spectrum Y^ζ over Y , define $f_g(\zeta)$ as the composite

$$R_n \otimes Y \xrightarrow{1 \otimes \Delta} R_n \otimes Y \otimes Y \xrightarrow{\psi^g \otimes \theta_g(\zeta)^{-1} \otimes 1} R_n \otimes R_n \otimes Y \xrightarrow{\mu \otimes 1} R_n \otimes Y$$

It follows quickly from this definition and the fact that $\psi^g \circ \delta = g \cdot \delta$ that this diagram commutes

$$\begin{array}{ccc} G \otimes R_n \otimes Y & \xrightarrow{\delta \otimes 1} & R_n \otimes Y \\ g \otimes f_g(\zeta) \downarrow & & \downarrow f_g(\zeta) \\ G \otimes R_n \otimes Y & \xrightarrow{\delta \otimes 1} & R_n \otimes Y. \end{array}$$

Proposition 4.13. Let Y^{G^ζ} be R_n -oriented with Thom class u (and associated Thom isomorphism T_u). Then the following diagram commutes:

$$\begin{array}{ccccc} R_n \otimes Y^{G^\zeta} & \xrightarrow{T_u} & G \otimes R_n \otimes Y & \xrightarrow{\delta \otimes 1} & R_n \otimes Y \\ \psi^g \otimes 1 \downarrow & & \downarrow 1 \otimes f_g(\zeta) & & \downarrow g^{-1} f_g(\zeta) \\ R_n \otimes Y^{G^\zeta} & \xrightarrow{T_u} & G \otimes R_n \otimes Y & \xrightarrow{\delta \otimes 1} & R_n \otimes Y, \end{array}$$

Proof. Expanding this out, we have

$$\begin{array}{ccccccc} R_n \otimes Y^{G^\zeta} & \xrightarrow{1 \otimes D} & R_n \otimes Y^{G^\zeta} \otimes Y & \xrightarrow{1 \otimes u \otimes 1} & R_n \otimes R_n \otimes G \otimes Y & \xrightarrow{\mu \otimes 1} & R_n \otimes G \otimes Y \xrightarrow{\delta \otimes 1} R_n \otimes Y \\ \psi^g \otimes 1 \downarrow & & & & & & \downarrow g^{-1} f_g(\zeta) \\ R_n \otimes Y^{G^\zeta} & \xrightarrow{1 \otimes D} & R_n \otimes Y^{G^\zeta} \otimes Y & \xrightarrow{1 \otimes u \otimes 1} & R_n \otimes R_n \otimes G \otimes Y & \xrightarrow{\mu \otimes 1} & R_n \otimes G \otimes Y \xrightarrow{\delta \otimes 1} R_n \otimes Y, \end{array}$$

which commutes by the definition of θ_g and the fact that ψ^g is a ring map.

□

Since the homotopy fibre of $\psi^g - 1 : R_n \rightarrow R_n$ is S , we see that $\text{hofib}((\psi^g - 1) \otimes 1) = Y^{G\zeta}$ in the diagram above. Since the horizontal maps are equivalences, we conclude:

Corollary 4.14. $\text{hofib}(g^{-1}f_g(\zeta) - 1) = \text{hofib}(f_g(\zeta) - g) = Y^{G\zeta}$.

We now focus on $X^{G\gamma}$. Define a map $\bar{z} : X \rightarrow R_n \otimes X$ by $\bar{z} = ((j-1) \otimes 1) \circ \Delta$. Since

$$\psi^g(j-1) = \theta_g(G\gamma) \cdot (j-1) = g\theta_g(\gamma) \cdot (j-1),$$

we see that this commutes

$$\begin{array}{ccccc} X & \xrightarrow{\Delta} & X \otimes X & \xrightarrow{(j-1) \otimes 1} & R_n \otimes X \\ \downarrow g\bar{z} & & \downarrow 1 \otimes \Delta & & \downarrow 1 \otimes \Delta \\ & & X \otimes X \otimes X & \xrightarrow{(j-1) \otimes 1} & R_n \otimes X \otimes X \\ & & \searrow g\theta_g(\gamma)(j-1) \otimes \theta_g(\gamma)^{-1} \otimes 1 & & \downarrow \psi^g \otimes \theta_g(\gamma)^{-1} \otimes 1 \\ R_n \otimes X & \xleftarrow{\mu \otimes 1} & R_n \otimes R_n \otimes X & & \end{array}$$

Passage along the right side of this diagram gives $f_g(\gamma) \circ \bar{z}$, so $f_g(\gamma) \circ \bar{z} = g\bar{z}$. Thus \bar{z} lifts to a map¹¹

$$z : X \rightarrow X^{G\gamma}.$$

which we will regard as the zero-section of the Thom spectrum.

The following result encourages us to regard $j-1$ as the Euler class of γ over X . Indeed, when $n = 1$, it is precisely the K-theoretic Euler class of the tautological line bundle over $\mathbb{C}P^\infty$. We continue to write $e^*(u) \in R_n^G(X^{G\gamma})$ for the Thom class of $X^{G\gamma}$, and $e^*(\bar{u}) \in R_n^0(X^{G\gamma})$ for its normalisation.

Proposition 4.15. $z^*(e^*(\bar{u})) = j-1$.

Proof. We first observe that one may derive $e^*(\bar{u})$ from the fibre sequence for $f_g(\gamma) - g$; that is, this commutes:

$$\begin{array}{ccccc} & & X^{G\gamma} & & \\ & \nearrow z & \downarrow & \searrow e^*(\bar{u}) & \\ X & \xrightarrow{\bar{z}} & R_n \otimes X & \xrightarrow{1 \otimes p} & R_n \otimes S = R_n \\ & & \downarrow f_g(\gamma) - g & & \\ & & R_n \otimes X & & \end{array}$$

Here $p : X \rightarrow S$ is induced by the projection of $K(\mathbb{Z}_p, n+1)$ to a point. The commutativity of the

¹¹The map z is, of course, not uniquely specified by this computation. However, any lift of \bar{z} will serve for our purposes, as will be evident from the proof of Proposition 4.15.

upper right triangle follows from the diagram of fiber inclusions

$$\begin{array}{ccccccc}
S \otimes X^{G\gamma} = X^{G\gamma} & \xrightarrow{\quad = \quad} & & & X^{G\gamma} \\
\eta \otimes 1 \downarrow & & & & \downarrow \\
R_n \otimes X^{G\gamma} & \xrightarrow{1 \otimes D} & R_n \otimes X^{G\gamma} \otimes X & \xrightarrow{1 \otimes e^*(u) \otimes 1} & R_n \otimes R_n \otimes G \otimes X & \xrightarrow{\mu \otimes 1} & R_n \otimes G \otimes X & \xrightarrow{\delta \otimes 1} & R_n \otimes X \\
& & & \searrow 1 \otimes e^*(u) & & & \downarrow 1 \otimes 1 \otimes p & & \downarrow 1 \otimes p \\
& & & & & & R_n \otimes G & \xrightarrow{\delta} & R_n,
\end{array}$$

since passage along the lower left defines $\delta e^*(u) = e^*(\overline{u})$. Then

$$z^*(e^*(\overline{u})) = (1 \otimes p) \circ \overline{z} = (1 \otimes p) \circ ((j-1) \otimes 1) \circ \Delta = j-1$$

□

Recall that $K^*(X) = K_*[[y]]$, and the natural transformation $t : R_n \rightarrow K$ of Proposition 3.21. We note that $t_*(e^*(u))$ is the K -Thom class of $X^{G\gamma}$. It satisfies $z^*(t_*(e^*(u))) = y$, for

$$z^*(e^*(t_*(u))) = t \circ (z^*(e^*(u))) = t \circ (j-1) = (1+y) - 1 = y.$$

We conclude:

Proposition 4.16. *The composite of the Thom isomorphism for $X^{G\gamma}$ and z^* , as a map from $K^*(X) \cong K^{*+2g(n)}(X^{G\gamma})$ to $K^{*+2g(n)}X$ is multiplication by y .*

Note that the image of z^* may be identified as the (split) subspace $yK_*[[y]] = \tilde{K}^*(K(\mathbb{Z}_p, n+1))$. Since z^* is evidently injective, we obtain:

Corollary 4.17. *The zero section restricts to a $K(n)$ -local equivalence $z : L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1) \rightarrow X^{G\gamma}$.*

5 Higher orientations for chromatic homotopy theory

5.1 n -orientations and formal group laws

The map $\rho : G \rightarrow X$ allows us to extend the notion of a complex orientation of a cohomology theory to the $K(n)$ -local category:

Definition 5.1. An n -orientation of a $K(n)$ -local ring spectrum E is a class $x \in E^G(X) = [X, G \otimes E]$ with the property that $\rho^*(x) \in E^G(G) = \pi_0 E = E_0$ is a unit.

We note that since $\rho_1 : L_{K(1)}S^2 \rightarrow X_1$ is the $K(1)$ -localisation of the inclusion $\mathbb{C}P^1 \subseteq \mathbb{C}P^\infty$, a 1-orientation is precisely a $K(1)$ -local complex orientation.

Examples 5.2. The following are n -oriented spectra:

1. The spectrum $R_n = X[\rho^{-1}]$ is naturally n -oriented. Define $x \in R_n^G(X)$ by

$$x = \delta^{-1} \otimes (j-1) : X \rightarrow G \otimes X[\rho^{-1}].$$

Then $\rho^*(x) = \delta^{-1} \cdot \rho^*(j-1) = \delta^{-1} \cdot \delta = 1$.

2. K is n -oriented, via the conveniently named class $x \in K^{2g(n)}(X) = K^G(X)$. The image of ρ in $K_*(X_n)$ is b_0 , and so $\rho^*(x) = x(b_0) = 1$. More generally, any power series $f(x) \in K_*[[x]]$ which begins with a unit multiple of x gives an n -orientation of K .
3. E_n is n -oriented by a lift of the previous orientation. More carefully,

$$E_n^G(X) \cong E_n^{2g(n)}(X) = u^{g(n)} \cdot E_n^0(X) = u^{g(n)} \cdot \mathbb{W}(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][\mathbb{Z}_p]$$

Then an orientation is given by the class of $u^{g(n)} \cdot g$, since the image of a fundamental class under ρ in $E_{n*}(X) \cong C(\mathbb{Z}_p, E_{n*})$ is a function which carries g to a unit.

4. MX_n is n -oriented via the map $x := (j^\gamma \otimes 1) \circ z$, as can be seen from the diagram:

$$\begin{array}{ccccc} X_n & \xrightarrow{z} & X^{G\gamma} & \xrightarrow{=} & X^\gamma \otimes G & \xrightarrow{j^\gamma \otimes 1} & MX_n \otimes G \\ \uparrow \rho & & & & \uparrow \eta \otimes 1 & & \\ G & \xleftarrow{=} & S \otimes G & & & & \end{array}$$

which commutes by the proof of Proposition 4.15. Thus $\rho^*(x) = j^\gamma \circ \eta$ is the unit of the ring spectrum MX_n .

Theorem 5.3. *An n -orientation x of E gives an isomorphism $E^\star(X) \cong E_\star[[x]]$, and the multiplication in X equips this ring with a formal group law $F(x, y) \in E_\star[[x, y]]$.*

Proof. We note that an n -orientation of E , $x \in E^G(X)$ defines, via the equivalence

$$z : L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1) \rightarrow X^{G\gamma},$$

a Thom class $u = (z^*)^{-1}(x) \in E^G(X^{G\gamma})$, since restricting u to each fibre gives the same class as restricting x to G , namely, a unit. Furthermore, extending z to all of $X = L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1) \vee S$, we see that the augmentation ideal of $E^\star X$ is $\tilde{E}^\star K(\mathbb{Z}_p, n+1) = z^* E^\star X^{G\gamma}$. Moreover, by the Thom isomorphism, the latter is the cyclic ideal generated by $z^* u = x$.

We conclude two things: that the augmentation ideal of $E^\star X$ is generated by x , and that it itself is isomorphic to $E^\star X^{G\gamma}$, and hence $E^\star X$ via the Thom isomorphism. We thereby inductively observe that the quotients of the filtration by powers of the augmentation ideal are free E^\star -modules of rank 1 generated by the powers of x , and so $E^\star X \cong E_\star[[x]]$.

The Künneth spectral sequence for $E^\star(X \otimes X)$ collapses to $E^\star(X_n) \otimes_{E_\star} E^\star(X_n) = E_\star[[x, y]]$, since each factor has free E^\star -cohomology. The properties of the formal group law all derive from the unital, associative multiplication on X .

□

5.2 A remark on n -gerbes

It is natural to ask what precisely an n -orientation is orienting. We recall that a complex orientation of E yields a theory of Chern classes for E in which the orientation is the first Chern class of the tautological bundle. Furthermore, the formal group law of the cohomology theory encodes the Chern class of a tensor product of line bundles for the cohomology theory.

We will consider a p -adic n -gerbe V over a space X to be a map $f_V : X \rightarrow K(\mathbb{Z}_p, n+1)$. Here we are purposefully confusing a gerbe with its Dixmier-Douady type of characteristic class. For an

n -oriented cohomology theory E , the orientation class x defines a first Chern class for V by the formula

$$c_1(V) := f_V^*(x) \in E^{S(\det)}(X).$$

The formal group law on E^\star then allows one to compute $c_1(V \otimes W) = F(c_1(V), c_1(W))$.

It is not apparent to the author how to extend this notion to higher rank (i.e., non-abelian) n -gerbes or higher Chern classes.

5.3 Multiplicative n -oriented spectra

Definition 5.4. An n -oriented spectrum R is *multiplicative* if there is a unit $t \in R_*$ such that the formal group law that R supports on $R^*(X_n)$ is given by the formula

$$F(x, y) = x + y + txy$$

Theorem 5.5. *The spectrum R_n with its natural n -orientation is the universal (i.e., initial) multiplicative n -oriented spectrum.*

Proof. This argument closely follows that of Spitzweck-Østvær [SØ09] for the motivic analogue of Snaith's theorem.

First, $X_n[\rho^{-1}]$ is multiplicative. We note that j is a map of ring spectra, as it is induced by the identity on X_n . Thus if m denotes multiplication in X_n ,

$$m^*(j) = j \otimes j \in R_n^*(X_n \otimes X_n) = R_{n*}[[x, y]],$$

since the tensor product is multiplication in the cohomology of the smash product. Definitionally, $j \otimes 1 = 1 + \delta x$, and $1 \otimes j = 1 + \delta y$, so $j \otimes j = 1 + \delta(x + y + \delta xy)$. Thus

$$F(x, y) = m^*(x) = m^*(\delta^{-1}(j - 1)) = \delta^{-1}(j \otimes j - 1) = x + y + \delta xy$$

Loosely, $R_n = X_n[\rho^{-1}]$ is universal because j is initial amongst maps from X_n to spectra in which ρ is invertible. More carefully, let E be an n -oriented, multiplicative spectrum, with orientation v , and whose formal group satisfies $F(v, w) = v + w + tvw$ for some $t \in E_G$. Then there is a map of ring spectra¹² $\phi : X_n \rightarrow E$ defined by $\phi = 1 + tv$. To check that ϕ is multiplicative, we need to see that $m^*(\phi) = \phi \otimes \phi \in E^\star(X_n \otimes X_n) = E^\star[[x, y]]$. But

$$m^*(\phi) = 1 + tm^*(v) = 1 + tF(v, w) = 1 + tv + tw + t^2vw = (1 + tv)(1 + tw) = \phi \otimes \phi.$$

Similarly, ϕ is unital, since t restricts to 0 over S^0 .

We note that

$$\phi_*(\rho) = (1 + tv) \circ \rho = tv \circ \rho = t \cdot (\rho^*v)$$

is a product of units, so ϕ extends over j to a map of ring spectra $\Phi : R_n = X_n[\rho^{-1}] \rightarrow E$.

Since E is n -oriented, the map $\delta : G \rightarrow X_n$ defines a function

$$\delta^* : E_\star[[v]]_G = E^G(X_n) \rightarrow E^G(G) = E_0$$

which carries a power series in v to the coefficient of v . Therefore $\phi \circ \delta = \delta^*(\phi) = t$. We see, then, that Φ is orientation preserving (i.e., $\Phi_*(x) = v$), since

$$v = t^{-1}(1 + tv - 1) = \Phi_*(\delta^{-1})(\phi - 1) = \Phi_*(\delta^{-1}(j - 1)) = \Phi_*(x).$$

¹²We are not claiming that this map is highly structured, only that it preserves multiplication and units up to homotopy.

Let $\Psi : X_n[\rho^{-1}] \rightarrow E$ is any other orientation-preserving map. Since it is orientation preserving, it must preserve the formal group law, so $\Psi_*(\delta) = t$. Consider the composite map $\psi := \Psi \circ j : X_n \rightarrow E$. Then

$$v = \Psi_*(x) = \Psi_*(\delta^{-1}(j - 1)) = t^{-1}(\psi - 1),$$

giving $\psi = 1 + tv = \phi$, and so $\Psi = \Phi$.

Therefore, for any multiplicative, n -oriented spectrum E , there exists a orientation-preserving map $\Psi : X_n[\rho^{-1}] \rightarrow E$, unique up to homotopy. \square

5.4 Identifying the coefficients of R_n

There is an “integral lift” $\mathbb{W}(K)$ of the cohomology theory K with homotopy groups

$$\mathbb{W}(K)_* = \mathbb{W}(\mathbb{F}_{p^n})[u^{\pm 1}]$$

One may define $\mathbb{W}(K)$ as the E_n -algebra

$$\mathbb{W}(K) := E_n / (u_1, \dots, u_{n-1}).$$

Note that reduction modulo p gives a natural transformation $\mathbb{W}(K) \rightarrow K$. Furthermore, the reduction map $E_n \rightarrow \mathbb{W}(K)$, being a ring homomorphism, carries an n -orientation of E_n to an n -orientation for $\mathbb{W}(K)$.

Proposition 5.6. *The formal group law on $\mathbb{W}(K)^{G^*}(X)$ is the universal multiplicative formal group law in the category of $\mathbb{W}(\mathbb{F}_{p^n})$ -algebras.*

Proof. K supports the multiplicative formal group over \mathbb{F}_{p^n} , via Proposition 3.2; $\mathbb{W}(K)$ is its universal deformation. But the universal deformation of the multiplicative group is the multiplicative group over $\mathbb{W}(\mathbb{F}_{p^n})$, which is clearly initial in the indicated category. \square

Recall that the Picard-graded homotopy of E_n associated to powers of G are

$$(E_n)_{G^*} = \mathbb{W}(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]] [s^{\pm 1}],$$

where $|s| = \dim(G) = 2g(n)$, following the discussion in Example 2.3. Thus $\mathbb{W}(K)_{G^*} = \mathbb{W}(\mathbb{F}_{p^n})[s^{\pm 1}]$.

We note that \mathbb{Z}_p^\times acts on the multiplicative group \mathbb{G}_m , and therefore on $\mathbb{W}(K)_{G^*}$, since the previous Proposition implies that $\mathbb{W}(K)_{G^*}$ co-represents \mathbb{G}_m in the category of $\mathbb{W}(\mathbb{F}_{p^n})$ -algebras. The action (through $\mathbb{W}(\mathbb{F}_{p^n})$ -algebra homomorphisms) is easily seen to be determined by the formula $\gamma \cdot s = \gamma s$.

For the next result, we recall that R_n is also equipped with an action of $\mathbb{Z}_p^\times = \mathbb{G}_n / S\mathbb{G}_n$:

Corollary 5.7. *As a \mathbb{Z}_p^\times -representation, $\pi_{G^*} R_n$ contains $\mathbb{Z}_p[\rho^{\pm 1}]$ as a split summand.*

Proof. We will write σ for a lift of a primitive $p^n - 1^{\text{st}}$ root of unity in \mathbb{F}_{p^n} to $\mathbb{W}(\mathbb{F}_{p^n})$, so that $\mathbb{W}(\mathbb{F}_{p^n}) = \mathbb{Z}_p[\sigma]$.

Since R_n is the universal multiplicative n -oriented spectrum, the multiplicativity of the orientation of $\mathbb{W}(K)$ gives us a unique oriented map $\Phi : R_n \rightarrow \mathbb{W}(K)$, which induces $\Phi_{G^*} : \pi_{G^*}(R_n) \rightarrow \pi_{G^*}(\mathbb{W}(K))$. The last Proposition gives us a unique oriented map $\Xi : \pi_{G^*} \mathbb{W}(K) \rightarrow \pi_{G^*} R_n[\sigma]$. It must then be the case that $\Phi_{G^*} \circ \Xi = \text{id}$ once we extend Φ_{G^*} over $\mathbb{W}(\mathbb{F}_{p^n}) = \mathbb{Z}_p[\sigma]$. Thus

$\pi_{G^*}(R_n)[\sigma]$ contains $\mathbb{W}(\mathbb{F}_{p^n})[\Xi(s)^{\pm 1}]$ as a split summand. As in the proof of Theorem 5.5, we must have $\Xi(s) = \rho$. As both maps were equivariant by universality properties, this splitting is equivariant.

We note that $\pi_0(S)$ does not contain any of the roots of unity in μ_{p^n-1} not lying in μ_{p-1} . If it contained one such τ , then for any $K(n)$ -local ring spectrum Y , $\pi_0(Y)$ would be a $\mathbb{Z}_p[\tau]$ -algebra. In particular, $\pi_0(K(n)) = \mathbb{F}_p$ would be an $\mathbb{F}_p[\tau] = \mathbb{Z}_p[\tau]/p$ -vector space, which is false if $\tau \notin \mu_{p-1} = \mathbb{F}_p^\times$.

Suppose now that $\tau \in \pi_0(R_n)$. Then the action of \mathbb{Z}_p^\times on $\mathbb{Z}_p[\tau] \subseteq \pi_0(R_n)$ is trivial since it is trivial upon extending further to $\mathbb{Z}_p[\sigma] = \mathbb{W}(K)_0$. Examine the long exact sequence

$$\cdots \longrightarrow \pi_0(S) \xrightarrow{\eta} \pi_0(R_n) \xrightarrow{\psi^g - 1} \pi_0(R_n) \longrightarrow \cdots$$

Then $\mathbb{Z}_p[\tau] \subseteq \ker(\psi^g - 1) \subseteq \pi_0(S)$, a contradiction.

Knowing that $\pi_{G^*}R_n[\sigma]$ contains $\mathbb{W}(\mathbb{F}_{p^n})[\rho^{\pm 1}]$ as a split summand, and that π_0R_n does not contain σ , we see that $\pi_{G^*}R_n$ contains $\mathbb{Z}_p[\rho^{\pm 1}]$ as a split summand. \square

5.5 R_n for large primes

Let $q := g^{p-1} = \zeta^{p-1}(1+p)^{p-1} = (1+p)^{p-1}$; q is a topological generator of $(1+p\mathbb{Z}_p)^\times$. Consequently, the equivalence $S = R_n^{h\mathbb{Z}_p^\times}$ may be factorised (following [Dav09]) as

$$S = (R_n^{h\mu_{p-1}})^{h(1+p\mathbb{Z}_p)^\times} = \text{hofib}(\psi^q - 1 : R_n^{h\mu_{p-1}} \rightarrow R_n^{h\mu_{p-1}})$$

A variation on a standard sparseness result for the Adams-Novikov spectral sequence (see, e.g., [GHM12]) yields the following:

Proposition 5.8. *If $n^2 < 2p - 3$, the long exact sequence in homotopy associated to*

$$S \xrightarrow{\eta} R_n^{h\mu_{p-1}} \xrightarrow{\psi^q - 1} R_n^{h\mu_{p-1}}$$

splits in a range of degrees, giving

$$\mathbb{Z}_p = \pi_0(S) \cong \pi_0(R_n^{h\mu_{p-1}}), \quad \text{and} \quad \pi_*(S) \cong \pi_*(R_n^{h\mu_{p-1}}) \oplus \pi_{*+1}(R_n^{h\mu_{p-1}})$$

*when $-2p + 1 \leq * < 0$.*

Proof. Equivalently, we may show these facts for $E_n^{h\mathbb{G}_n^1}$. The homotopy of this spectrum is computed via the spectral sequence

$$H^s(\mathbb{G}_n^1, (E_n)_t) \implies \pi_{t-s}(E_n^{h\mathbb{G}_n^1}).$$

Since p is odd, our assumption implies that $(p-1)$ and p do not divide n , and thus \mathbb{G}_n is a p -adic analytic Lie group of dimension n^2 with no p -torsion; thus, its cohomological dimension is $n^2 + 1$ (see [Mor85]). Similarly, \mathbb{G}_n^1 has cohomological dimension n^2 . So the only contribution to π_* comes from $H^s(\mathbb{G}_n^1, (E_n)_{s+*})$ where $0 \leq s \leq n^2$.

However, one may compute this group cohomology as

$$H^s(\mathbb{G}_n^1/\mu_{p-1}, (E_n)_{s+*}^{\mu_{p-1}}) = H^s(S\mathbb{G}_n, (E_n)_{s+*}^{\mu_{p-1}}),$$

which vanishes when $s + *$ is not a multiple of $2(p-1)$. Assuming that $n^2 < 2(p-1)$ and that $-2(p-1) < * \leq 0$ ensures that $-2(p-1) < * \leq s + * \leq s \leq n^2 < 2(p-1)$, so the only possible contribution is when $s = -*$ (so $s + * = 0$). So for $* \leq 0$,

$$\pi_*(R_n^{h\mu_{p-1}}) = H^{-*}(\mathbb{G}_n^1, (E_n)_0)$$

The same analysis holds (for $n^2 + 1 < 2(p-1)$) to show that when $-2(p-1) < * \leq 0$, $\pi_*(S) = H^{-*}(\mathbb{G}_n, (E_n)_0)$.

Now, since $p \nmid n$, the reduced determinant is split, giving an isomorphism $\mathbb{G}_n \cong \mathbb{G}_n^1 \times \mathbb{Z}_p$, and the \mathbb{Z}_p factor acts trivially on $(E_n)_0$ (and hence $H^*(\mathbb{G}_n^1, (E_n)_0)$). Thus the (collapsing) Lyndon-Hochschild-Serre spectral sequence gives $H^0(\mathbb{G}_n, (E_n)_0) \cong H^0(\mathbb{G}_n^1, (E_n)_0)$, and if $* < 0$,

$$\begin{aligned} H^{-*}(\mathbb{G}_n, (E_n)_0) &\cong H^1(\mathbb{Z}_p, H^{-1-*}(\mathbb{G}_n^1, (E_n)_0)) \oplus H^0(\mathbb{Z}_p, H^{-*}(\mathbb{G}_n^1, (E_n)_0)) \\ &\cong H^{-1-*}(\mathbb{G}_n^1, (E_n)_0) \oplus H^{-*}(\mathbb{G}_n^1, (E_n)_0) \end{aligned}$$

□

This implies that for $0 \leq m \leq 2p-1$,

$$1 = \psi^q : \pi_{-m} R_n^{h\mu_{p-1}} \rightarrow \pi_{-m} R_n^{h\mu_{p-1}},$$

since the unit map is surjective in homotopy there. Now, the G -periodicity of $\pi_\star(R_n)$ yields a $G^{\otimes p-1}$ -periodicity of $\pi_\star(R_n^{h\mu_{p-1}})$, so

$$[G^{\otimes k(p-1)}, \Sigma^m R_n^{h\mu_{p-1}}] \cong \pi_{-m}(R_n^{h\mu_{p-1}})$$

Corollary 5.9. *If $n^2 < 2p-3$, and $0 \leq m \leq 2p-1$, the endomorphism ψ^q of $[G^{\otimes k(p-1)}, \Sigma^m R_n^{h\mu_{p-1}}]$ is multiplication by q^k .*

Proof. It suffices to observe that for $f \in \pi_{-m}(R_n^{h\mu_{p-1}})$ this commutes:

$$\begin{array}{ccccc} \Sigma^{-m} G^{\otimes k(p-1)} = G^{\otimes k(p-1)} \otimes S^{-m} & \xrightarrow{1 \otimes f} & G^{\otimes k(p-1)} \otimes R_n^{h\mu_{p-1}} & \xrightarrow{\delta^{k(p-1)}} & R_n^{h\mu_{p-1}} \\ & \searrow q^k \otimes f & \downarrow g^{k(p-1)} \otimes \psi^q & & \downarrow \psi^q \\ & & G^{\otimes k(p-1)} \otimes R_n^{h\mu_{p-1}} & \xrightarrow{\delta^{k(p-1)}} & R_n^{h\mu_{p-1}} \end{array}$$

□

5.6 Analogues of the image of J

Corollary 5.10. *Let $k \in \mathbb{Z}$, and write $k = p^s m$, where m is coprime to p . Then $[G^{\otimes k(p-1)}, S^1]$ contains a subgroup isomorphic to \mathbb{Z}/p^{s+1} . Furthermore, if $n^2 < 2p-3$, there is an exact sequence*

$$0 \rightarrow \mathbb{Z}/p^{s+1} \rightarrow [G^{\otimes k(p-1)}, S^1] \rightarrow N_{s+1} \rightarrow 0$$

where $N_{s+1} \leq \pi_{-1}(S)$ is the subgroup of p^{s+1} -torsion elements.

Proof. Without assumptions on n , Corollary 5.7 implies that $[G^{\otimes i}, R_n]$ has a split summand \mathbb{Z}_p upon which the action of ψ^g is through multiplication by g^i . Taking $i = k(p-1)$, we conclude that the action of $\psi^g = \psi^{g^{p-1}}$ on the corresponding summand $\mathbb{Z}_p \leq [G^{\otimes k(p-1)}, R_n^{h\mu_{p-1}}] \cong \pi_0(R_n)$ is by multiplication by q^k .

Consider the long exact sequence obtained by applying $[G^{\otimes k(p-1)}, -]$ to the fibre sequence

$$\dots \longrightarrow R_n^{h\mu_{p-1}} \xrightarrow{\psi^g - 1} R_n^{h\mu_{p-1}} \longrightarrow S^1 \longrightarrow \Sigma R_n^{h\mu_{p-1}} \xrightarrow{\psi^g - 1} \Sigma R_n^{h\mu_{p-1}} \longrightarrow \dots$$

The first map contains a factor which is given by multiplication by $q^k - 1$ on \mathbb{Z}_p . Since $q^k - 1$ generates the procyclic subgroup $p^{s+1}\mathbb{Z}_p \subseteq \mathbb{Z}_p$, we see that $[G^{\otimes k(p-1)}, S^1]$ contains the indicated cokernel as a subgroup.

Now, if $n^2 < 2p - 3$, the results of the previous section give us a very precise computation of this sequence:

$$\mathbb{Z}_p \xrightarrow{\psi^g - 1} \mathbb{Z}_p \longrightarrow [G^{\otimes k(p-1)}, S^1] \longrightarrow [G^{\otimes k(p-1)}, \Sigma R_n^{h\mu_{p-1}}] \xrightarrow{\psi^g - 1} [G^{\otimes k(p-1)}, \Sigma R_n^{h\mu_{p-1}}],$$

and Corollary 5.9 implies that $\psi^g - 1$ is again multiplication by a unit multiple of p^{s+1} . Lastly, the identification $[G^{\otimes k(p-1)}, \Sigma R_n^{h\mu_{p-1}}] \cong \pi_{-1}(R_n^{h\mu_{p-1}})$ as a summand of $\pi_{-1}(S)$ (with complement \mathbb{Z}_p) yields the desired short exact sequence. \square

6 Redshift

We will write $M(p)$ for the $K(n)$ -local mod p Moore spectrum; i.e., the cofibre of the map $p : S \rightarrow S$. When $p > 2$, $M(p)$ is a ring spectrum, homotopy associative if $p > 3$; we will assume the latter throughout this section.

Proposition 6.1. *There is a map $v : S\langle \det \rangle \rightarrow Z \otimes M(p)$ which induces an isomorphism in E_n after multiplying by the identity on $M(p)$:*

$$v_* : E_{n*}(S\langle \det \rangle \otimes M(p)) \xrightarrow{\cong} E_{n*}(Z \otimes M(p)) .$$

Thus v induces an equivalence $S\langle \det \rangle \otimes M(p) \simeq Z \otimes M(p)$.

Proof. Notice that after smashing with $M(p)$, the element $g \in \mathbb{Z}_p \subseteq [X_n, X_n]$ becomes homotopic to $\zeta \in \mathbb{F}_p \subseteq [X_n \otimes M(p), X_n \otimes M(p)]$. Thus the indicated lift in the following diagram exists:

$$\begin{array}{ccc} S\langle \det \rangle & \xrightarrow{\quad v \quad} & Z \otimes M(p) \\ \delta \downarrow & & \downarrow \alpha \otimes 1 \\ R_n & \xrightarrow{1 \otimes \eta} & R_n \otimes M(p) \\ \psi^g - g \downarrow & & \downarrow (\psi^g - \zeta) \otimes 1 \\ R_n & \xrightarrow{1 \otimes \eta} & R_n \otimes M(p) \end{array}$$

To see that v_* is an isomorphism, we note that the image in

$$E_{n*}(R_n \otimes M(p)) = C(\mathbb{Z}_p, E_{n*}/p) = C(\mathbb{Z}_p, \mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]])[u^{\pm 1}])$$

of $\alpha \otimes 1$ and $(1 \otimes \eta) \circ \delta$ are both generated by the function $f_0 : \mathbb{Z}_p \rightarrow \mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$ given by $f_0(x) = x \bmod p$. \square

This fact allows us to multiply any element of the Picard-graded homotopy of $Y \otimes M(p)$ with v , for any spectrum Y .

Proposition 6.2. *The map $\beta_n^{p-1} : S\langle \det \rangle^{\otimes p-1} \rightarrow L_{K(n)}K(E_{n-1})$ is, after smashing with $M(p)$, homotopic to multiplication of the unit $\eta : S \rightarrow L_{K(n)}K(E_{n-1})$ by v^{p-1} .*

Proof. The statement of the proposition amounts to the claim that the following diagram commutes:

$$\begin{array}{ccccc} S\langle \det \rangle^{\otimes p-1} & \xrightarrow{\rho_n^{p-1}} & X_n & \xrightarrow{i \circ B\varphi_{n-1}} & L_{K(n)}K(E_{n-1}) \\ \downarrow v^{\otimes p-1} & & \downarrow 1 \otimes \eta & & \downarrow 1 \otimes \eta \\ S \otimes M(p) & \xrightarrow{\eta \otimes 1} & X_n \otimes M(p) & \xrightarrow{i \circ B\varphi_{n-1} \otimes 1} & L_{K(n)}K(E_{n-1}) \otimes M(p), \end{array}$$

since the lower composite is the unit of $L_{K(n)}K(E_{n-1})$ tensored with the identity on $M(p)$. The first square commutes by construction, and the second is evident. \square

Corollary 6.3. *Multiplication by $\beta_n \in \pi_{S\langle \det \rangle} L_{K(n)}K(E_{n-1})$ is an equivalence.*

Proof. Since $v^{p-1} : S\langle \det \rangle^{\otimes p-1} \otimes M(p) \rightarrow Z^{\otimes p-1} \otimes M(p) = M(p)$ is an equivalence by Proposition 6.1, smashing it with the identity of $L_{K(n)}K(E_{n-1})$ is also an equivalence. By the previous Proposition, then multiplication by β_n^{p-1} :

$$[\beta_n^{p-1}] = [v^{p-1}] : (E_n)_\star(K(E_{n-1}) \otimes M(p)) \rightarrow (E_n)_{\star+S\langle \det \rangle^{\otimes p-1}}(K(E_{n-1}) \otimes M(p))$$

is an isomorphism. However, this implies that β_n^{p-1} is an isomorphism in K_* and hence an equivalence, since K_n is a module spectrum for $E_n \otimes M(p)$, and hence in the same Bousfield class. \square

One may go slightly further. One of the chromatic redshift conjectures of [AR06] (specifically Conjecture 4.4) proposes an equivalence

$$L_{K(n)}K(\Omega_{n-1}) \simeq E_n,$$

where Ω_{n-1} is a suitable interpretation of the algebraic closure of the fraction field of E_{n-1} . In particular, this should restrict to a unital E_∞ map $r_n : L_{K(n)}K(E_{n-1}) \rightarrow E_n$. When $n = 1$, this is a map $L_{K(1)}K(\mathbb{Q}_p) \rightarrow KU_p^\wedge$ corresponding to a choice of embedding $\mathbb{Q}_p \rightarrow \mathbb{C}$. Let us presume only the existence of such a map r_n . The previous results indicate the commutativity of the solid part of the following diagram:

$$\begin{array}{ccccc} L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)_+ & \xrightarrow{B\varphi_{n-1}} & L_{K(n)}\Sigma^\infty BGL_1(E_{n-1})_+ & \xrightarrow{i} & L_{K(n)}K(E_{n-1}) \\ \downarrow & & \nearrow & & \downarrow \text{---} r_n \\ R_n & \xrightarrow{\simeq} & E_n^{hSG_n} & \xrightarrow{\quad} & E_n \end{array}$$

Now, the path along the lower left is the map φ_n . The path along the upper right (including the conjectural redshift map r_n) is an E_∞ map, and thus by Theorem 3.19, differs from φ_n by at worst an Adams operation. So up to a suitable re-embedding of $E_n^{hS\mathbb{G}_n}$ into E_n , the diagram must commute if r_n is to exist. This provides a falsifiable condition upon any candidate redshift map r_n : it must give a factorisation of the homotopy fixed point inclusion $E_n^{hS\mathbb{G}_n} \rightarrow E_n$.

7 Questions

This section is purely speculative, and is intended as a collection of vaguely posed questions which the reader is invited to either clarify or (more likely) disprove.

Perhaps the most evident thing lacking in this paper is a geometric definition of the cohomology theory R_n . Despite all of the analogies with K-theory given above, we do not have a description of $R_n^*(X)$ akin to the Grothendieck group of (some generalisation of) vector bundles over X . The discussion of orientations on p -adic n -gerbes indicates a possible direction with this question, but it is not apparent to the author what the corresponding analogue of higher rank vector bundles should be (although the work of Michael Murray and others on bundle gerbe modules [Mur10, BCM⁺02] may point a way forward).

A related deficiency is the fact that we have described a $K(n)$ -local analogue of the *image* of the J-homomorphism, but not the homomorphism itself.¹³ Classically, one can think of the J-homomorphism as built from either the functions $O(m) \rightarrow \Omega^m S^m$, or (in families, after delooping) the function that assigns to a vector bundle the associated sphere bundle. Neither of these has an obvious analogue in our construction. Such a description would be enlightening. In connection with the first question this raises the obvious question: does an n -bundle gerbe have a geometrically defined “sphere” bundle, where $S\langle\det\rangle$ is our replacement for the spherical fibre?

This, in turn, raises the question of whether there is a good unstable description of the map $\rho_n : S\langle\det\rangle \rightarrow L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)_+$ analogous to the inclusion $\mathbb{C}P^1 \rightarrow \mathbb{C}P^\infty$. It follows from [Bou01] that $S\langle\det\rangle$ is the $K(n)$ -localisation of a suspension spectrum, but a more geometric description of this spectrum and map is desirable.

Further, the definition of an n -oriented spectrum raises the question of whether it is possible to repeat the whole program of analysing complex-oriented spectra by their formal group laws, but in the $K(n)$ -local category, with $K(\mathbb{Z}_p, n+1)$ replacing $\mathbb{C}P^\infty$. All of our examples, however, have resulted in essentially multiplicative formal groups. One standout is the spectrum MX_n for which we haven’t even the beginnings of a computation of the associated formal group when $n > 1$. We would like to believe that it plays a universal role analogous to MU in the complex-oriented case, but do not have any evidence to back this up.

Lastly, we have no examples in hand of n -oriented spectra whose associated formal group law is *additive* (i.e., the analogue of singular homology). Does such a theory exist? If so, and if it could be made part of an Atiyah-Hirzebruch spectral sequence for $K(n)$ -local theories, the rather convoluted construction of the formal group law in Theorem 5.3 (requiring the entirety of section 4) could be vastly simplified.

¹³Of course, one may give such a homomorphism as the map $\gamma : \Omega^\infty R_n\langle 1 \rangle \rightarrow BGL_1 S$; this then reiterates our desire for a more geometric definition of R_n .

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